UNIFORM WEYL ASYMPTOTICS FOR OFF-DIAGONAL SPECTRAL PROJECTIONS

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A dissertation submitted to Johns Hopkins University in conformity
with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland
July 2020

Abstract

Let (M, g) denote a smooth, compact Riemannian manifold, without boundary of dimension $n \geq 2$. Furthermore, we assume that all points on the manifold are non self-focal. In our paper we provide a new proof for the off-diagonal behavior of the Schwartz kernel for the spectral projection operator onto the unit frequency interval $(\lambda, \lambda+1]$ as $\lambda \to \infty$; we denote this kernel by $K_\lambda(x, y) = K(x, y; \lambda)$. By using standard techniques from harmonic and microlocal analysis we manage to obtain a result similar to [2]. However, whereas their result requires that x and y be taken in an ever-shrinking neighborhood of the diagonal, we prove that the same asymptotics can be found over a uniformly-small neighborhood, dependent only on the manifold (M, g).

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Acknowledgments

I am eternally grateful to everyone who has made my dissertation and defense possible. To my advisor, Chris Sogge, for your guidance, generosity, and patience. Thank you for working so closely with me and for supporting my education all these years. To my undergraduate mentor, Dan-Andrei Geba, for your enduring faith and friendship. To my mother, for always believing in me. You have supported my passion for math my entire life. You've helped me keep my eyes on the goal, and were always there to cheer me on. To my wife, Samantha, for your unending love, patience, and emotional support over this past decade. You helped keep me grounded and always showed me unconditional love, through thick and thin. And finally, to my many friends and colleagues I've made along the way. In particular, thank you to Rich Brown for helping me become the best teacher I could be, and to Sabrina Raymond for always being a friend I could count on to help me navigate the ups and downs of graduate life at Hopkins. I am indebted to each and every one of you, and owe you all so much more than I will ever be able to repay. You have all shaped me in ways for the better, and I am grateful to have had you there beside me throughout this journey.

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Introduction

1.1 Statement of the Problem and Previous Results

Let (M, g) denote a smooth, compact Riemannian manifold without boundary. A standard result shows that $-\Delta_g$ is a positive, self-adjoint operator where Δ_g is the negative Laplace-Beltrami operator defined on (M, g), a smooth, compact Riemannian manifold of dimension $n \geq 2$ without boundary. As outlined in sources like [8], if $t > 0$, the *heat operator* $e^{-t\Delta_g} : H_0(M) \to H_1(M)$ is a continuous operator between Sobolev spaces due to its smoothing properties. Then, since the inclusion $H_1(M) \hookrightarrow H_0(M) = L^2(M)$ is compact by the Rellich-Kondrachov compactness theorem [3], this means $e^{-t\Delta_g} : L^2(M) \to L^2(M)$ is also a compact operator, and hence admits an othonormal basis of eigenfunctions for $L^2(M)$, with a discrete spectrum of eigenvalues given by $\{e^{-t\mu_j}\}_{j=0}^{\infty}$, each with finite multiplicity and accumulating only at zero. It quickly follows that the same set of eigenfunctions is also an orthonormal basis for $-\Delta_g$, but with the set of non-negative eigenvalues $\{\mu_j\}_{j=0}^{\infty}$.

In our paper we will be focusing on the *half-wave* operator $e^{it\sqrt{-\Delta_g}}$. Seeing as $\mu_j \geq 0$, it will be convenient to refer to the eigenvalues for $-\Delta_g$ as $\mu_j = \lambda_j^2$, with $\lambda_j \in \mathbb{R}^+$. In particular, we see that $\lambda_0 = 0$ corresponds to the eigenspace of constant functions (which are annihilated by $-\Delta_g$) and that since the set of eigenvalues is unbounded, we have the increasing sequence $\{\lambda_j\}_{j=0}^{\infty}$ with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$, counted with multiplicity, where $\lambda_j \to \infty$ as $j \to \infty$.

To this spectrum we associate an orthonormal basis of L^2-normalized eigenfunctions, $\{e_j\}_{j=0}^{\infty}$.

That is,

$$-\Delta_g e_j(x) = \lambda_j^2 e_j(x), \quad \text{with } \langle e_j, e_k \rangle = \int_M e_j(y)\overline{e_k(y)}\, dV_g = \delta_k^j, \text{ for all } j,k = 0,1,2,\dots$$

For each λ_j we will denote its corresponding eigenspace by

$$V_{\lambda_j} = \{\phi \in C^2(M) : -\Delta_g \phi = \lambda_j^2 \phi\}.$$

The spectral theory then assures us that

$$(-\Delta_g f)(x) = \sum_{j=0}^{\infty} \lambda_j^2 (E_j f)(x), \quad I = \sum_{j=0}^{\infty} E_j, \tag{1.1.1}$$

for all $f \in L^2(M)$, where $(E_j f)(x) = \langle f, e_j \rangle e_j(x) = \left(\int_M f(y)\overline{e_j(y)}\, dV_g(y)\right) e_j(x)$ denotes the orthogonal projection of $f(x)$ onto the span of $e_j(x)$. Therefore

$$\sum_{\{j\,:\,\lambda_j = \lambda\}} E_j : L^2(M) \to V_\lambda$$

must be the orthogonal projection onto the eigenspace V_λ. As is common with the study of Weyl asymptotics, we will be looking at the partial-projection operators S_λ, defined by

$$\begin{aligned} S_\lambda : L^2(M) &\to \bigoplus_{\lambda_j \in [0,\lambda]} V_{\lambda_j} \\ f(x) &\mapsto (S_\lambda f)(x) := \sum_{\{j\,:\,\lambda_j \le \lambda\}} (E_j f)(x). \end{aligned} \tag{1.1.2}$$

Let $K(x,y;\lambda) := K_\lambda(x,y)$ denote the integral kernel of the difference $S_{\lambda+1} - S_\lambda$, defined on $M \times M$. It is the goal of this paper to prove that if all of the points on M are non self-focal, then this kernel exhibits $o(\lambda^{n-1})$ behavior, uniform in x and y, given that x, y are in a sufficiently small neighborhood of each other. More precisely, we wish to prove the following theorem:

Theorem 1.1. *Let (M,g) be a compact, smooth, Riemannian manifold of dimension $n \ge 2$, without boundary, and suppose that all $x \in M$ are non self-focal. Let $K_\lambda(x,y)$ denote the integral kernel of the operator $S_{\lambda+1} - S_\lambda$. Then for all $\varepsilon > 0$ there exists $\delta = \delta_{M,\varepsilon} > 0$ and*

$\Lambda > 1$ *such that*

$$\sup_{\substack{x,y\in M\\ d_g(x,y)<\delta}} \left| K_\lambda(x,y) - \frac{\lambda^{n-1}}{(2\pi)^{\frac{n}{2}}} \frac{J_{\frac{n-2}{2}}\Big(\lambda\, d_g(x,y)\Big)}{(\lambda\, d_g(x,y))^{\frac{n-2}{2}}} \right| \lesssim \varepsilon\lambda^{n-1}, \tag{1.1.3}$$

for all $\lambda > \Lambda$, where J_α is the Bessel function of the first kind with index α and $d_g(x,y)$ is the Riemannian distance between x and y.

Please note that here $f(\lambda) \lesssim g(\lambda)$ means that there exists some $C > 0$, independent of λ (or ε), such that $f(\lambda) \leq Cg(\lambda)$.

Our result is in contrast to the main results of Canzani and Hanin in [2], presented here:

Theorem 1.2 (Theorem 2 [2])**.** *Let (M,g) be a compact, smooth, Riemannian manifold of dimension $n \geq 2$, without boundary. Suppose that $x_0 \in M$ is non self-focal, and consider any nonnegative function $r(\lambda)$ satisfying $r(\lambda) \to 0$ as $\lambda \to \infty$. Then,*

$$\sup_{x,y\in B(x_0,r(\lambda))} |R(x,y;\lambda)| = o(\lambda^{n-1}) \tag{1.1.4}$$

as $\lambda \to \infty$. Here $B(x_0,r(\lambda))$ denotes the geodesic ball of radius $r(\lambda)$ centered at x_0 and the rate of convergence depends on x_0 and r_λ.

If the Riemannian distance between x and y is less than the injectivity radius of (M,g), then [2] use the inverse of the exponential map to write $K_\lambda(x,y)$ as

$$K_\lambda(x,y) = \frac{\lambda^n}{(2\pi)^n} \int_{|\xi|_{g_y}<1} e^{i\lambda\langle exp_y^{-1}(x),\xi\rangle_{g_y}} \frac{d\xi}{\sqrt{|g_y|}} + R(x,y;\lambda), \tag{1.1.5}$$

where the remainder $R(x,y;\lambda)$ is a smooth function of x and y. A direct corollary of this theorem is the following:

Corollary 1.3 (Theorem 3 [2])**.** *Let (M,g) be a compact, smooth, Riemannian manifold of dimension $n \geq 2$, without boundary. Suppose that $x_0 \in M$ is non self-focal, and consider any nonnegative function $r(\lambda)$ satisfying $r(\lambda) \to 0$ as $\lambda \to \infty$. Then,*

$$\sup_{x,y\in B(x_0,r(\lambda))} \left| K_\lambda(x,y) - \frac{\lambda^{n-1}}{(2\pi)^{\frac{n}{2}}} \frac{J_{\frac{n-2}{2}}\Big(\lambda\, d_g(x,y)\Big)}{(\lambda\, d_g(x,y))^{\frac{n-2}{2}}} \right| = o(\lambda^{n-1}), \tag{1.1.6}$$

where J_α is the Bessel function of the first kind with index α; $d_g(x,y)$ is the Riemannian distance between x and y; and $B(x_0, r(\lambda))$ is the geodesic ball of radius $r(\lambda)$ centered at x_0.

The formulation of this corollary forms the basis for the way we've stated our main result. Hence, when moving forward, we will primarily be referring to this version of the main result form [2].

As mentioned in [2], these results stand apart from other estimates of $R(x, y; \lambda)$ as $\lambda \to \infty$ in that it brings together similar results for both diagonal ($x = y$) and off-diagonal ($x \neq y$) proved by others. Specifically, [13] and [9] show that if M consists of only non self-focal points, then $R(x, x; \lambda) = o(\lambda^{n-1})$ when x lives in a compact subset of the diagonal in $M \times M$. [9] also derived the same results for $R(x, y; \lambda)$, assuming that (x, y) lives in a compact subset of $M \times M$ supported *away* from the diagonal, with the added assumptions that x and y are mutually nonfocal, and that at least one of them is also non self-focal. In contrast to [2], however, by assuming that all of the points in M are non self-focal, we are able to strengthen the $o(\lambda^{n-1})$ bounds to be uniform in x and y, rather than requiring x and y to be in a ball with radius $r(\lambda)$ shrinking to zero.

In [4], it was shown that there exists some $\varepsilon > 0$ for which if $d_g(x, y) < \varepsilon$, then

$$K_\lambda(x, y) = \frac{\lambda^n}{(2\pi)^n} \int_{|\xi|_{g_y} < 1} e^{i\lambda\psi(x,y,\xi)} \frac{d\xi}{\sqrt{|g_y|}} + O(\lambda^{n-1}), \tag{1.1.7}$$

where the phase function satisfies

$$\psi(x, y, \xi) = \langle x - y, \xi \rangle + O(|x - y|^2 |\xi|).$$

In particular, this phase function is a solutions to certain Hamilton-Jacobi equations which cannot be described explicitly. [2] remarks that this makes direct analysis of the off-diagonal behavior difficult, and so they choose to opt for a modified phase function. Specifically, they use a form of the parametrix for the half-wave operator, e^{-itP}, also used in [18]. In this construction the authors exchange the geometric phase function of the Hörmander parametrix for a phase function of the form

$$\phi(t, x, y, \xi) = \langle \exp_y^{-1}(x), \xi \rangle - t|\xi|_{g_y}.$$

In doing so, they were able to extract the off-diagonal behavior of $K_\lambda(x, y)$ by using the results of [6], in which the behavior of Fourier integral operators with global phase functions is analyzed.

Our paper, on the other hand, achieves the main result using only standard harmonic and microlocal techniques, like those found in [11] and [12], and the usual Hadamard parametrix, similar to [5].

1.2 Notation and Definitions

We will now briefly outline some of the common notation and definitions that you'll encounter in this paper.

1.2.1 Riemannian Geometry

Throughout our paper, all Riemannian manifolds (M, g) we consider will be of dimension $n \geq 2$, compact, smooth, and without boundary. The notation and derivations in this section are standard and can be found in references such as [11]. Working locally, $g = (g_{ij}(x))$ denotes the Riemannian metric, with $g^{-1} = (g_{ij}(x))^{-1} =: (g^{ij}(x))$ denoting the inverse metric, and $|g| := \det g$. In these local coordinates we can express the Laplace-Beltrami operator as

$$\Delta_g = |g|^{-1/2} \sum_{j,k=0}^{n} \partial_j (|g|^{1/2} g^{jk}) \partial_k, \tag{1.2.1}$$

and denote it's principal symbol by

$$p(x, \xi) = \sum_{j,k} g^{jk}(x) \xi_j \xi_k. \tag{1.2.2}$$

We then calculate the Hamiltonian vector field associated with the energy function $\tilde{p} = \frac{1}{2} p$,

$$H_{\tilde{p}} = \frac{\partial \tilde{p}}{\partial \xi} \frac{\partial}{\partial x} - \frac{\partial \tilde{p}}{\partial x} \frac{\partial}{\partial \xi},$$

which gives rise to the *cogeodesic flow* on $T^*M \setminus 0$, namely $\Phi_t = \exp t H_{\tilde{p}}$, where

$$\Phi_t(x_0, \xi_0) = (x(t), \xi(t))), \quad \text{with } (x_0, \xi_0) = (x(0), \xi(0)).$$

To find geodesics on M we simply need to project the flow lines in T^*M onto M. Hence the projection $\Pi\Phi_t : T^*M \to M$, given by $(x(t), \xi(t)) \mapsto x(t)$, provides us with the unit speed geodesics on (M, g) we seek. Moreover, having fixed some $x_0 \in M$, it is always possible to find a set of local coordinates on T^*M such that

$$\Pi\Phi_t(x_0,\xi) = \Pi\Phi_1(x_0,t\xi) = t\xi.$$

Therefore it will be convenient to consider the restriction of the flow to the unit cosphere bundle, $\Phi_t : S^*M \to S^*M$, where $S^*M \subset T^*M$ is given by

$$S^*M = \{(x,\xi) \in T^*M : p(x,\xi) = ||\xi||_{g^{-1}(x)} = 1\}. \tag{1.2.3}$$

Of particular interest to us is the following set:

Definition 1.4 (Geodesic loop)**.** Suppose a geodesic $\gamma(t)$ starts at the point $x_0 \in M$ with initial direction $\xi_0 \in S^*_{x_0}M = \{\xi : (x_0,\xi) \in S^*M\}$. This geodesic is called a *geodesic loop through* x_0 if there exists some time $0 < |T| < \infty$ for which $\gamma(0) = \gamma(T) = x_0$.

Note that for each cotangent vector $\xi \in T^*_xM$ the corresponding tangent vector, $v \in T_xM$, will be given by the *musical isomorphism* $v_j = \sum_i g^{ij}(x)\xi_i$ [11]. Because of this we will often refer to cotangent vectors as *"directions"* by a slight abuse of notation.

Since every geodesic loop can be identified with its initial unit tangent direction, we can therefore find a correspondence between the set of geodesic loops through x and the following subset of S^*M:

$$\mathcal{L}_x = \big\{\xi \in S^*_xM : \Phi_T(x,\xi) = (x,\eta), \text{ for some } |T| > 0 \text{ and } \eta \in S^*_xM\big\}. \tag{1.2.4}$$

We will referrer to $\mathcal{L}_x$ as the set of *looping directions*, since it comprises the cotangent directions that give rise to geodesic loops. Obviously geodesic loops on our manifold need not loop back smoothly, i.e. $\eta \neq \xi$. If they do, then we refer to them as *periodic geodesics*:

Definition 1.5 (Periodic geodesic)**.** If $\gamma(t)$ is the geodesic loop corresponding to $\xi \in \mathcal{L}_x$, and $\Phi_T(x,\xi) = (x,\xi)$, we say that γ is a *periodic geodesic with period* T.

Lastly, we need to define what it means for a point in our manifold to be *non self-focal*:

Definition 1.6 (Non self-focal point)**.** A point $x \in M$ is said to be *non self-focal* if $\mathcal{L}_x$ has zero measure with respect to the induced measure on S^*_xM. In this case we write $|\mathcal{L}_x| = 0$.

Working locally in our coordinate patch $\Omega \subseteq M$, we have that $S^*M \simeq \tilde{\Omega} \times \mathbb{S}^{n-1}$, and so the measure

$|\mathcal{L}_x|$ is understood as that of the sphere, $\mathbb{S}^{n-1}$.

Because of the assumptions needed for our main result, we will assume that none of the points on M are self-focal. Examples of such manifolds include any $n \geq 3$ dimensional compact manifold with nonpositive curvature [13] [1], and the flat torus $\mathbb{T}^2$. These are in stark contrast to the manifold of $\mathbb{S}^{n-1}$, where every direction at every point on the sphere is a looping direction. In other words $\mathcal{L}_x$ has full measure everywhere on the sphere. Moreover, every loop is periodic with the same period. Since the wave front set for the half-wave operator propagates along geodesics, our proof will fail on manifolds like $\mathbb{S}^{n-1}$ where geodesic flow can return to itself over highly concentrated regions.

1.2.2 Wave Front Sets of Distributions

Suppose that $u \in \mathcal{D}'(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open set. Recall that the singular support of u, denoted $\operatorname{sing\,supp} u$, is the subset of Ω defined by the following condition: $x_0 \notin \operatorname{sing\,supp} u \iff \exists \mathcal{N}_{x_0}$, an open neighborhood of x_0, and $f \in C^\infty(\Omega)$, for which $u(\phi) = \int_\Omega f(x)\phi(x)dx$ for all $\phi \in C_0^\infty(\mathcal{N}_{x_0})$. In other words, x_0 is not in the singular support if u restricted to a small neighborhood of x_0 is a smooth function.

Recall that the Paley-Wiener-Schwartz theorem states if $u \in \mathcal{E}'(\mathbb{R}^n)$, the space of compactly-supported distributions, then $\widehat{u} \in C^\omega(\mathbb{C}^n)$, the space of entire functions on $\mathbb{C}^n$. Moreover, $u \in C_0^\infty(\mathbb{R}^n) \iff \widehat{u}(\xi)$ is rapidly decreasing, meaning

$$|\widehat{u}(\xi)| \leq C_N(1+|\xi|)^{-N}, \quad \text{for all } N \in \mathbb{N}.$$

However, even if $u \notin C_0^\infty$, it still is possible that $\widehat{u}(\xi)$ is rapidly decreasing in *some* directions, but not all. This provides the motivation for defining the *wave front set* of a distribution.

Definition 1.7 (Conic neighborhood)**.** We say that an open set $\mathcal{N}$ is a *conic neighborhood* of $\Sigma \subset \mathbb{R}^n \setminus 0$ if $\Sigma \subseteq \mathcal{N}$ and whenever $\xi \in \mathcal{N}$, then $\lambda\xi \in \mathcal{N}$ for all $\lambda > 0$.

While we tend to think of $\operatorname{sing\,supp} u$ as consisting of the *locations* of the singularities of u, we now use this idea of conic neighborhoods to construct an analogous set consisting of the *directions* of the singularities. We let $u \in \mathcal{E}'(\mathbb{R}^n)$ and define the set $\Gamma(u)$, as before, as a complement: for all $\eta \in \mathbb{R}^n \setminus 0$ we have $\eta \notin \Gamma(u) \iff \exists \mathcal{N}$, a conic neighborhood of η, in which $\widehat{u}(\xi)$ is rapidly decreasing for all $\xi \in \mathcal{N}$. While $\Gamma(u)$ helps us localize the singular directions of a distribution $u \in \mathcal{E}'(\mathbb{R}^n)$, we need to do a little more to extend this idea to a general distribution in $\mathcal{D}'(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$ open.

A straightforward lemma from [11] shows that if $\rho \in C_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{E}'(\mathbb{R}^n)$, then $\Gamma(\rho u) \subseteq \Gamma(u)$. In other words, localizing the distribution u localizes the set of singular directions. With this lemma we are finally able to get a handle on the defining the wave front set for an arbitrary distribution.

Let $u \in \mathcal{D}'(\Omega)$ as before. Thus if $\rho \in C_0^\infty(\mathbb{R}^n)$ then $\rho u \in \mathcal{E}'(\mathbb{R}^n)$ and so $\Gamma(\rho u)$ is well-defined. For each $x \in \Omega$ we then construct the following set:

$$\Gamma_x(u) := \bigcap_{\{\rho \in C_0^\infty \,:\, \rho(x) \neq 0\}} \Gamma(\rho u). \tag{1.2.5}$$

Hence $\Gamma_x(u)$ finds the singular directions of u at the point x, and it follows that u is smooth in a neighborhood of x if and only if $\Gamma_x(u) = \varnothing$. Lastly, we define the wave front set as follows:

Definition 1.8 (Wave front set)**.** If $u \in \mathcal{D}'(\Omega)$, with Ω open, then the *wave front set* of u is defined as

$$WF(u) = \{(x, \xi) \in \Omega \times \mathbb{R}^n \setminus 0 \,:\, \xi \in \Gamma_x(u)\}. \tag{1.2.6}$$

Wave front sets the fundamental objects of study in mircolocal analysis, as they encompass both the singular points of distributions — note that the projection of $WF(u)$ onto Ω is exactly $\operatorname{sing\,supp} u$ — a well as the directions with which the singularities can propagate. Moreover, unlike $\Gamma(u)$, $WF(u)$ is invariant under a change of variables (cf. theorem 0.4.6 [12]):

Theorem 1.9. *Suppose that $\kappa : X \to Y$ is a diffeomorphism between open subsets of $\mathbb{R}^n$. For any $\Lambda \subseteq Y \times \mathbb{R}^n \setminus 0$ define the pullback of Λ by κ as*

$$\kappa^*\Lambda := \left\{(x, \xi) \,:\, \left(\kappa(x), (\kappa')^{-\intercal}\xi\right) \in \Lambda\right\}.$$

Then

$$WF(\kappa^* u) = \kappa^* WF(u), \quad u \in \mathcal{D}'(Y). \tag{1.2.7}$$

Of particular interest to us is the 1-parameter family of distributions

$$e^{itP}(x, y) \in \mathcal{D}'(\Omega \times \mathbb{R} \times \Omega),$$

the Schwartz kernel of the half-wave operator e^{itP}, where $P = \sqrt{-\Delta_g}$ is the Laplace-Beltrami operator defined on a compact manifold (M, g), with Ω an open coordinate patch of M (see below). As outlined in [12], the wave front sets of the these distributions evolve according to the (co)geodesic flow on the manifold,

$$WF\left(e^{itP}\right) = \{(x, t, y, \xi, \tau, \eta) : \Phi_t(y, -\eta) = (x, \xi), \tau = p(x, \xi)\} \tag{1.2.8}$$

where $p(x, \xi)$ is the principal symbol of $P = \sqrt{-\Delta_g}$, and Φ_t as above. This key fact will prove very important later in the proof of our main theorem.

1.2.3 Functional Calculus

Suppose that P is a linear operator with an orthonormal basis of eigenvectors, $\{e_j(x)\}_{j=0}^{\infty}$, with corresponding eigenvalues $\{\mu_j\}_{j=0}^{\infty}$. The Borel functional calculus gives us a way to define new operators as functions of P, based on its spectrum. Namely if $m(\tau)$ is a Borel function on $\mathbb{R}$, then we may define the new operator

$$m(P) := \sum_{j=0}^{\infty} m(\mu_j) E_j \tag{1.2.9}$$

by the method of eigenfunction expansion. The projection onto the eigenspace V_λ mentioned earlier,

$$\sum_{\{j \,:\, \lambda_j = \lambda\}} E_j : L^2(M) \to V_\lambda,$$

is an example of one of these spectral operators where $m(\tau) = \delta_\lambda^\tau$, the Kronecker delta function. We are particularly interested in the case when $m(\tau) = \sqrt{\tau}$, and $P = -\Delta_g$ with eigenfunctions $\{\lambda_j^2\}_{j=0}^{\infty}$. It is easy to see that $m(P) = \sqrt{-\Delta_g}$ has the same collection of eigenfunctions as $-\Delta_g$, but with eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$, and that $\sqrt{-\Delta_g}(\sqrt{-\Delta_g} f) = -\Delta_g f$ for all $f \in L^2(M)$.

If we further assume that $m(\tau) \in L^2(\mathbb{R})$ then we may use the Fourier inversion formula to write an alternate expression for $m(P)$ involving $\widehat{m}(t)$:

$$
\begin{aligned}
(m(P)f)(x) &= \sum_{j=0}^{\infty} m(\mu_j)(E_j f)(x) \\
&= \frac{1}{2\pi} \sum_{j=0}^{\infty} \left(\int_{-\infty}^{\infty} e^{it\mu_j} \widehat{m}(t)\, dt \right) (E_j f)(x) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{m}(t) \left(\sum_{j=0}^{\infty} e^{it\mu_j} (E_j f)(x) \right) dt \\
&=: \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{m}(t)(e^{itP} f)(x)\, dt,
\end{aligned} \tag{1.2.10}
$$

where $\exp(itP) = e^{itP}$ is a 1-parameter family of (unitary) operators (if P is self-adjoint) with $t \in \mathbb{R}$. In particular, if $P = \sqrt{-\Delta_g}$, as above, then we have the half-wave operator as mentioned before. Since we are dealing exclusively with the half-wave operator from here on, all instances of P should be understood to mean $\sqrt{-\Delta_g}$ unless stated otherwise.

To study the operators associated with orthogonal projections onto a single eigenspace V_λ would be rather difficult considering that $\mathrm{supp}_\tau \delta_\lambda^\tau$ has measure zero, and so any methods involving integration will be unable to differentiate it from the zero function. Instead, we "stretch out" our Borel function into the characteristic function of the interval $(\lambda, \lambda+1]$, rather than of a single point.

Working on (M, g) as before, recall that the operator S_λ is the orthogonal projection of an $L^2(M)$ function onto the set of frequencies in the range $[0, \lambda]$, which of course corresponds to the characteristic function $m(\tau) = \chi_{[0,\lambda]}(\tau)$. Therefore, if we wish to project onto the unit-length band of frequencies, $(\lambda, \lambda+1]$, we only need to consider the operator $S_{(\lambda,\lambda+1]} := S_{\lambda+1} - S_\lambda$, which we can rewrite as an integral operator, whose kernel we will denote $K_\lambda(x,y)$:

$$\begin{aligned}(S_{(\lambda,\lambda+1]}f)(x) &= \sum_{\{j:\lambda_j\le\lambda+1\}}(E_jf)(x) - \sum_{\{j:\lambda_j\le\lambda\}}(E_jf)(x)\\ &= \sum_{\{j\,:\,\lambda<\lambda_j\le\lambda+1\}}(E_jf)(x)\\ &= \sum_{\{j\,:\,\lambda<\lambda_j\le\lambda+1\}}\left(\int_M f(y)\overline{e_j(y)}\,dV_g(y)\right)e_j(x)\\ &= \int_M\left(\sum_{\{j\,:\,\lambda<\lambda_j\le\lambda+1\}}e_j(x)\overline{e_j(y)}\right)f(y)\,dV_g(y) \qquad (1.2.11)\\ &= \int_M K_\lambda(x,y)f(y)\,dV_g(y).\end{aligned}$$

The λ-asymptotic behavior of the kernel $K_\lambda(x,y)$ is the focus of theorem 1.1.

If we denote $\chi_\lambda := \chi_{(\lambda,\lambda+1]} = \chi_{[0,\lambda+1]} - \chi_{[0,\lambda]}$, then

$$S_{(\lambda,\lambda+1]} = \sum_{j=0}^{\infty}\chi_\lambda(\lambda_j)E_j =: \chi_\lambda(P).$$

with $\chi_\lambda(\tau)\in L^2(\mathbb{R})$, meaning we can apply (1.2.10) to find an alternate expression for $K_\lambda(x,y)$:

$$S_{(\lambda,\lambda+1]}f = (\chi_\lambda(P)f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\chi_\lambda}(t)\left(e^{itP}f\right)(x)\,dt. \qquad (1.2.12)$$

Note that $\chi_{-1/2} = \mathbb{1}_{(-1/2,1/2]}$ is an even function, and hence has a real Fourier transform:

$$\begin{aligned}\widehat{\chi_{-1/2}}(t) &= \int_{-\infty}^{\infty}e^{-it\tau}\chi_{-1/2}(\tau)\,d\tau\\ &= \int_{-1/2}^{1/2}\cos(t\tau) - i\sin(t\tau)\,d\tau\\ &= \frac{\sin(t/2)}{t/2}. \qquad (1.2.13)\end{aligned}$$

This means that $\chi_\lambda(\tau) = \chi_{-1/2}\big(\tau - (\lambda+\frac{1}{2})\big)$ has Fourier transform

$$\begin{aligned}\widehat{\chi_\lambda}(t) &= e^{-it(\lambda+1/2)}\,\widehat{\chi_{-1/2}}(t)\\ &= e^{-it(\lambda+1/2)}\,\frac{\sin(t/2)}{t/2}. \qquad (1.2.14)\end{aligned}$$

Hence, by combining (1.2.12) and (1.2.14), we have

$$
\begin{aligned}
(S_{(\lambda,\lambda+1]}f)(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi_\lambda(t)(e^{itP}f)(x)\,dt \\
&= (2\pi)^{-1}\int_{\infty}^{\infty}\left(e^{-it(\lambda+1/2)}\,\frac{\sin(t/2)}{t/2}\sum_{j=0}^{\infty}e^{it\lambda_j}\int_M f(y)e_j(x)\overline{e_j(y)}\,dV_g(y)\right)dt \\
&= \int_M\left((2\pi)^{-1}\int_{\infty}^{\infty}\frac{\sin(t/2)}{t/2}e^{-it(\lambda+1/2)}e^{itP}(x,y)\,dt\right)f(y)\,dV_g(y), \qquad (1.2.15)
\end{aligned}
$$

where

$$
e^{itP}(x,y) = \sum_{j=0}^{\infty}e^{it\lambda_j}e_j(x)\overline{e_j(y)}, \qquad (1.2.16)
$$

is the integral kernel of the half-wave operator, $e^{it\sqrt{-\Delta_g}}$. Since (1.2.11) and (1.2.15) both describe the same operator, we conclude that

$$
\boxed{K_\lambda(x,y) = (2\pi)^{-1}\int_{-\infty}^{\infty}\frac{\sin(t/2)}{t/2}e^{-it(\lambda+1/2)}e^{itP}(x,y)\,dt. \qquad (1.2.17)}
$$

With $K_\lambda(x,y)$ in its final form, we are now ready to outline the techniques used to prove our main result.

2

Outline of Proof

Our main result says that the difference between $K_\lambda(x,y)$ and

$$J_\lambda(x,y) := \frac{\lambda^{n-1}}{(2\pi)^{\frac{n}{2}}} \frac{J_{\frac{n-2}{2}}\Big(\lambda\, d_g(x,y))\Big)}{(\lambda\, d_g(x,y))^{\frac{n-2}{2}}} \tag{2.0.1}$$

is asymptotically $o(\lambda^{n-1})$, so long as x and y are sufficiently close. To prove this, we begin by picking an arbitrary $\varepsilon > 0$, and requiring that x and y be contained in some small open chart on M, say Ω_α. Working locally, we will be able to find $\delta_\alpha = \delta_{M,\varepsilon,\alpha} > 0$ and $\Lambda_\alpha \gg 1$ for which

$$\sup_{\substack{x,y\in\Omega_\alpha \\ d_g(x,y)<\delta_\alpha}} |K_\lambda(x,y) - J_\lambda(x,y)| \lesssim \varepsilon\lambda^{n-1}, \quad \forall\lambda \geq \Lambda_\alpha.$$

Thus $\{\Omega_\alpha\}_\alpha$ will form an open cover of (M,g), which will admit a finite subcover $\{\Omega_{\alpha_i}\}_{i=1}^N$ since M is compact. Therefore if we take

$$\delta := \min\{\delta_{\alpha_1}, \ldots, \delta_{\alpha_N}\}, \quad \Lambda := \max\{\Lambda_{\alpha_1}, \ldots, \Lambda_{\alpha_N}\}$$

we will have our uniform bounds over all of M, as desired. Therefore it suffices to do all of our computations locally in an arbitrary coordinate patch.

To begin, we will assume that our δ is at least as small as the injectivity radius of (M,g), and that $\Lambda > 1$. As we progress through the rest of the proof, we may continue to make δ smaller and Λ larger as needed. A key part of the proof will investigate sets of geodesics that flow from a fixed point $x_0 \in M$, for some fixed, finite amount of time $|t| < \infty$. In order to get $o(\lambda^{n-1})$ bounds, or better, we end up breaking the integral in (1.2.17) into three pieces: a "small time" integral, with compact support on a neighborhood of $t = 0$; a "medium time" integral, with compact support away from from zero; and a "large time" integral, with unbounded support away from zero.

To do this we define a bump function, $\beta(t) \in C_0^\infty(\mathbb{R})$, with the following properties:

- $\beta(t)$ is an even function,
- $\beta(t) \equiv 1$ for $|t| \leq 1/2$,
- $\beta(t) \equiv 0$ for $|t| > 1$.

With these properties, for any $T > 1$ we have the following identity:

$$\begin{aligned} 1 &\equiv \beta(t) + (1 - \beta(t)) \\ &= \beta(t) + (1 - \beta(t))\Big(\beta(t/T) + (1 - \beta(t/T))\Big) \\ &= \beta(t) + (1 - \beta(t))\beta(t/T) + (1 - \beta(t))(1 - \beta(t/T)) \\ &= \beta(t) + (1 - \beta(t))\beta(t/T) + (1 - \beta(t/T)). \end{aligned}$$

Using this identity we are now able to decompose the kernel $K_\lambda(x, y)$ as

$$K_\lambda(x, y) = K_1(x, y) + K_2(x, y) + K_3(x, y),$$

where

$$K_1(x, y) = (2\pi)^{-1} \int_{-\infty}^{\infty} \beta(t)\, \frac{\sin(t/2)}{t/2}\, e^{-it(\lambda+1/2)}\, e^{itP}(x, y)\, dt \tag{2.0.2a}$$

$$K_2(x, y) = (2\pi)^{-1} \int_{-\infty}^{\infty} (1 - \beta(t))\, \beta(t/T)\, \frac{\sin(t/2)}{t/2}\, e^{-it(\lambda+1/2)}\, e^{itP}(x, y)\, dt \tag{2.0.2b}$$

$$K_3(x, y) = (2\pi)^{-1} \int_{-\infty}^{\infty} (1 - \beta(t/T))\, \frac{\sin(t/2)}{t/2}\, e^{-it(\lambda+1/2)}\, e^{itP}(x, y)\, dt, \tag{2.0.2c}$$

Note that with these three integrals (2.0.2a) matches our notion of an integrand with "small time" support, while (2.0.2b) and (2.0.2c) represent our "medium time" and "large time" integrands respectively.

By decomposing the integral kernel in this way, we are able to break up our proof into three more manageable pieces, each of which requires its own unique approach. By using the triangle inequality we see that

$$|K_\lambda(x,y) - J_\lambda(x,y)| \leq |K_1(x,y) - J_\lambda(x,y)| + |K_2(x,y)| + |K_3(x,y)|,$$

meaning that our desired result will follow if we can show that each of the terms on the right are individually $o(\lambda^{n-1})$ or better.

Since our proof will consist of three separate parts, each will be outlined in its own subsection below for ease of readability. Moreover, with $\varepsilon > 0$ fixed, as above, we will find an appropriate $\delta_* > 0$ and $\Lambda_* > 1$ for each part separately. To finish the proof then, we then simply require that $\delta = \min\{\delta_{\text{small}}, \delta_{\text{med}}, \delta_{\text{large}}\}$ and $\Lambda = \max\{\Lambda_{\text{small}}, \Lambda_{\text{med}}, \Lambda_{\text{large}}\}$.

2.1 Small Time Estimates

Proving $o(\lambda^{n-1})$ bounds for the first term, $|K_1(x,y) - J_\lambda(x,y)|$, involves fairly standard techniques. To begin we use Euler's identity to rewrite the operator $e^{itP}(x,y)$ in (2.0.2a) as an expression involving $\cos(tP)(x,y)$:

$$\begin{aligned} e^{itP}(x,y) &= (e^{itP}(x,y) + e^{-itP}(x,y)) - e^{-itP}(x,y) \\ &= 2\cos(tP)(x,y) - e^{-itP}(x,y), \end{aligned}$$

where

$$\cos(tP)(x,y) = \sum_{j=0}^{\infty} \cos(t\lambda_j) e_j(x)\overline{e_j(y)} \tag{2.1.1}$$

is the integral kernel of the operator $\cos(tP)$. This allows us express our kernel K_1 as

$$K_1(x,y) = K_{1,a}(x,y) + K_{1,b}(x,y),$$

where

$$K_{1,a}(x,y) = \pi^{-1} \int_{-\infty}^{\infty} \beta(t)\, \frac{\sin(t/2)}{t/2}\, e^{-it(\lambda+1/2)} \cos(tP)(x,y)\, dt, \tag{2.1.2a}$$

$$K_{1,b}(x,y) = (2\pi)^{-1} \int_{-\infty}^{\infty} \beta(t)\, \frac{\sin(t/2)}{t/2}\, e^{-it(\lambda+1/2)} e^{-itP}(x,y)\, dt. \tag{2.1.2b}$$

And since $K_{1,b}(x, y)$ is simply the Fourier transform of a compactly supported smooth function in t, and P a positive operator, then it will be rapidly decreasing in λ, and hence is $O(\lambda^{-N+n})$ for any $N > 1$. This means that the contribution from $|K_{1,b}|$ will be negligible when compared with the rest of the $o(\lambda^{n-1})$ bounds.

This leaves us needing to control $|K_{1,a}(x, y) - J_\lambda(x, y)|$, which we can do by using the Hadamard parametrix for $\cos(tP)$. Using the standard asymptotic expansion for the parametrix, we are able to express $K_{1,a}$ as the sum

$$K_{1,a}(x, y) = \sum_{\nu=0}^{N} K_{1,\nu}(x, y) + K_{1,R}(x, y), \tag{2.1.3}$$

for any $N > n + 3$. From the way the parametrix is constructed we quickly see that the only term from this expansion that will contribute significantly to our computation is the top-order term, $K_{1,0}(x, y)$. Hence our desired bounds will follow if we can show that

$$|K_{1,0}(x, y) - J_\lambda(x, y)| \lesssim \varepsilon\lambda^{n-1}.$$

Our next step involves introducing a smooth cutoff function $\Psi(s)$, which is identically 1 on a small neighborhood of $s = 1$, and is compactly supported away from zero. By replacing the term $\cos(t|\xi|)$, found in $K_{1,0}(x, y)$, with $\cos(t|\xi|)\Psi(|\xi|/\lambda)$, we arrive at a new kernel $K_\Psi(x, y)$, which differs from $K_{1,0}$ in that the integral has compact ξ-support. Because of the properties we chose for $\Psi(s)$, this concentrates the ξ-support near $|\xi| \approx \lambda$ and away from $|\xi| = 0$. Applying the triangle inequality again leave us with two terms,

$$|K_{1,0}(x, y) - K_\Psi(x, y)| \quad \text{and} \quad |K_\Psi(x, y) - J_\lambda(x, y)|.$$

To handle $K_{1,0} - K_\Psi$ we first integrate in t, which gives us a function in ξ that is rapidly decreasing and is $O\big((1 + |\lambda - |\xi||)^{-N}\big)$. After switching to polar coordinates, and using the fact that $1 - \Psi(|\xi|/\lambda)$ is identically zero on a neighborhood of $|\xi| = \lambda$, we end up with an integral that is $O(\lambda^{-N+n})$ for any $N > 1$, again meaning that it will not contribute to our little oh bound.

This leaves us to handle the term $K_\Psi - J_\lambda$. Up until now, every kernel we've worked with has involved a t-integral that was localized near $t = 0$ by way of the cutoff function $\beta(t)$. To finish our proof of the small time estimates, we now introduce one last kernel, which we call $I_\lambda(x,y)$, which agrees with $K_\psi(x,y)$ except for the lack of $\beta(t)$. By removing the cutoff function, we are able to directly compute the t-integral as a Fourier transform. Using this result, along with the fact that $\Psi(|\xi|/\lambda) \equiv 1$ on a neighborhood of $|\xi| = \lambda$, it follows that

$$I_\lambda(x,y) \approx \int_{|\xi|\in[\lambda,\lambda+1]} e^{iz\cdot\xi}\, d\xi,$$

where $|z| = d_g(x,y)$. Note that this is essentially the Fourier transform the the characteristic function of the annulus, $\lambda \le |\xi| \le \lambda + 1$.

Moreover, as found in sources like [16], [7], and [12], the Fourier transform of the spherical Lebesgue measure, $d\sigma(\omega)$, is given by the following radial function:

$$\int_{\mathbb{S}^{n-1}} e^{iz\cdot\omega}\, d\sigma(\omega) = (2\pi)^{n/2} \frac{J_{\frac{n-2}{2}}(|z|)}{|z|^{\frac{n-2}{2}}}. \tag{2.1.4}$$

This means that

$$J_\lambda(x,y) \approx \int_{\mathbb{S}^{n-1}} e^{iz\cdot\omega}\, d\sigma(\omega),$$

where $|z| = d_g(x,y)$, and so $I_\lambda - J_\lambda$ is essentially measuring how much these two transforms differ from each other. After proving a small lemma, we quickly find that

$$|I_\lambda(x,y) - J_\lambda(x,y)| \lesssim d_g(x,y)\lambda^{n-1} + \lambda^{n-2},$$

which means that so long as $d_g(x,y) < \delta \le \varepsilon$, then we will have $|I_\lambda - J_\lambda| = o(\lambda^{n-1})$ as desired.

Therefore, the only thing holding us back from finishing our small time proof is the difference $|K_\Psi - I_\lambda|$. Recall that the t-integral in K_Ψ has compact support, whereas I_λ does not. They both, however, have compact support in their ξ-integral. By changing to polar coordinates, and applying a standard technique from stationary phase arguments, we are able to show that

$$|K_\Psi(x,y) - I_\lambda(x,y)| \lesssim \lambda^{-N+n} \int_{-\infty}^{\infty} \frac{1-\beta(t)}{|t|^N}\, dt = O(\lambda^{-N+n}),$$

for any $N > 1$, and hence it does not contribute to our little oh asymptotics. It's worth noting here that we made use of the fact that $1 - \beta(t) \equiv 0$ on a small neighborhood of $t = 0$ to eliminate the singularity in the integral above. This concludes our proof of the small times estimates by requiring that $\delta_{\text{small}} = \varepsilon$ and Λ_{small} be large enough so that each of our $O(\lambda^{-N+n})$ inequalities hold.

2.2 Medium Time Estimates

The proofs used in this section borrow largely from similar techniques found in section 5.4 of [11]. To begin we first fix some $x_0 \in \tilde{\Omega}$, which by our hypothesis must be non self-focal. Next we define a length functional, $L(x,\xi) : S^*M \to (0, +\infty]$, where $L(x,\xi) = |t|$ is defined to be the smallest time $t \neq 0$ for which $\Phi_t(x,\xi) = (x,\eta)$, for some $\eta \in S_x^*M$. If no such t exists, then we set $L(x,\xi) = +\infty$. This means that the set of looping directions at x, which we denoted $\mathcal{L}_x$ is equivalent to

$$\mathcal{L}_x = \{\xi \in S_x^*M \,:\, L(x,\xi) < \infty\}.$$

Similarly, for some fixed $T > 0$, we denote the set of directions that loop back in time T or less by

$$\mathcal{L}_{x,T} = \{\xi \in S_x^*M \,:\, L(x,\xi) \leq T\}.$$

By the lower semi-continuity of $L(x,\xi)$ [Sog14] assures us that $\mathcal{L}_{x_0,T}$ is closed. Therefore the compliment of these "bad directions" must be open in $\mathbb{S}^{n-1}$, and hence any "good direction" — one that loops back in time greater than T — must be contained in some small open neighborhood inside of our coordinate patch on T^*M. In other words, if $y \in M$ is close enough to x_0, and $\eta \in \mathbb{S}^{n-1}$ is close enough to a "good direction", say ξ_0, then $L(y,\eta) > T$.

Next, since x_0 is non self-focal, $\mathcal{L}_{x_0,T}$ must have measure zero, and so we can construct open sets $\mathcal{V}_0, \mathcal{V}_1 \subseteq S_{x_0}^*M$ such that

$$\mathcal{L}_{x_0,T} \subset \mathcal{V}_0 \Subset \mathcal{V}_1,$$

where $|\mathcal{V}_1| \approx T^{-1} < \varepsilon$. Since

$$\mathcal{V}_1^{\complement} \Subset \mathcal{V}_0^{\complement} \subset \mathcal{L}_{x_0,T}^{\complement},$$

then $\xi \in \mathcal{V}_0^{\complement} \implies L(x_0,\xi) > T$. Hence for each such ξ we can find $\mathcal{W}_\xi$, an open neighborhood of (x_0,ξ), giving us $\{\mathcal{W}_\xi\}_{\xi \in \mathcal{V}_0^{\complement}}$ as an open cover of $\mathcal{V}_0^{\complement}$, which is compact. We then take a finite subcover and find the intersections of all projections onto M — call this $\mathcal{U}$ — and the union of all projections onto $T_{x_0}^*M$ — call this $\mathcal{V}$. This means that $x_0 \in \mathcal{U}$, $\mathcal{V}_0^{\complement} \subseteq \mathcal{V}$, and for all $y \in \mathcal{U}$ and all

$\eta \in \mathcal{V}$ we have $L(y,\eta) > T$.

We now pick some $\delta_* > 0$ (smaller than the injectivity radius) small enough that $B_{\delta_*}(x_0) \Subset \mathcal{U}$. This means that if $d_g(x_0, y) < \delta_*$, and $\eta \in \mathcal{V}_0^{\complement}$ then $L(y,\eta) > 0$. From here we introduce two separate cutoff functions: one on $\tilde{\Omega}$, and one on $\mathbb{S}^{n-1}$. We construct $\sigma \in C^\infty(\mathbb{S}^{n-1})$ to be identically 1 on $\mathcal{V}_1^{\complement}$, and identically 0 on $\overline{\mathcal{V}_0}$; while $\psi \in C_0^\infty(\tilde{\Omega})$ is supported inside of $B_{\delta_*}(x_0)$ and is identically 1 on $B_{\delta_*/2}(x_0)$.

We use these functions to create a pair of complementary pseudodifferential operators $B(x,D)$, $b(x,D)$ such that

$$B(x,D) + b(x,D) = \psi(x),$$

where $\psi(x)$ is the operator that is multiplication by ψ. The key takeaway is that since $\psi(x) = \psi(y) = 1$ for any $x, y \in B_{\delta_*/2}(x_0)$, then

$$\begin{aligned} e^{itP}(x,y) &= \left(\psi \circ e^{itP}\psi^*\right)(x,y) \\ &= \left(\left(B + b\right) \circ e^{itP} \circ \left(B^* + b^*\right)\right)(x,y). \end{aligned}$$

After expanding this conjugation we are able to rewrite $K_2(x,y)$ as the sum of four separate kernels, three of which involve $b(x,D)$, plus one that is equal to $B(x,D) \circ e^{itP} \circ B^*(x,D)$.

A simple lemma borrowed from [11] proves that these first three kernels are $\varepsilon\lambda^{n-1} + O(\lambda^{n-2})$, which comes about directly from the fact that $|\mathcal{V}_1| \lesssim \varepsilon$. We then prove another lemma which implies that geodesic flow from $(x,\xi) \in \operatorname{supp} B$ will avoid any other $(y,\eta) \in \operatorname{supp} B$ by some fixed minimum distance $\delta_T > 0$, depending only on T. It then follows from propagation of singularities [12] [11] that since geodesic flow from $\operatorname{supp} B$ avoids itself, we must have that $B \circ e^{itP} \circ B^*$ is a smooth kernel, and thus contributes only negligible terms to our λ asymptotics.

Hence there must exist some $\Lambda_{\text{med}} \gg 1$, $\delta_{\text{med}} > 0$, and $T \gg 1$ such that if $d_g(x,y) < \delta_{\text{med}}$ and $\lambda \geq \Lambda_{\text{med}}$, then all four parts of $K_2(x,y)$ will be $\varepsilon\lambda^{n-1} + O(\lambda^{n-2})$ as desired, concluding our proof for the medium time kernel.

2.3 Large Time Estimates

We come now to the final part of the proof of our main theorem. Since the definition of $K_3(x, y)$ involves the expression $(1 - \beta(t/T))\frac{\sin(t/2)}{t/2} \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, we are able to express the t-integral as a Fourier transform, which we denote $\widehat{X_T}$. This allows us to rewrite K_3 as

$$K_3(x, y) = \sum_{j=0}^{\infty} \widehat{X_T}\left(\lambda - \lambda_j + \tfrac{1}{2}\right) e_j(x)\overline{e_j(y)},$$

where, by using Euler's formula to convert $\sin(t/2)$ into its complex exponential form, we can show that $\widehat{X_T}(\theta)$ is proportional to the oscillatory integral

$$I(\theta) := \int_{-\infty}^{\infty} \left(\frac{1 - \beta(s)}{s}\right) e^{-is\theta}\, ds,$$

with $\theta \approx \lambda - \lambda_j$. A short proof shows that $I(\theta) = O((1 + |\theta|)^{-N})$ for any $N \in \mathbb{N}$, meaning that $\widehat{X_T}$ is rapidly decreasing.

In order to prove that the series above is convergent, and has the desired behavior in λ, we first apply the Cauchy-Schwarz inequality to separate the series into a product of series each involving only either $e_j(x)$ or $e_j(y)$. Using the rapid decay of $\widehat{X_T}$ allows us to compare each of these series to the following double sum

$$\sum_{k=0}^{\infty} \sum_{\lambda_j \in I_k} (1 + T|\lambda - \lambda_j|)^{-N} |e_j(z)|^2,$$

where $|I_k| \approx T^{-1}$ and $\bigcup_{k=0}^{\infty} I_k = [0, \infty)$. From here we borrow a lemma from [11] (cf. 5.4.1) which tells us that

$$\sum_{\lambda_k \in [\lambda, \lambda + T^{-1}]} |e_k(z)|^2 = O(T^{-1}\lambda^{n-1}),$$

for all $\lambda > \Lambda_*$ and $d_g(z, x_0) < \delta_*$ where x_0 is any non self-focal point and $T \gg 1$. Applying the lemma directly to our series allows us to show that

$$|K_3(x, y)| \lesssim T^{-1} \int_0^{\infty} (1 + k)^{-N} (\lambda + k/T)^{n-1}\, dk,$$

for $\lambda > \Lambda_*$ and $d_g(x, y) < \delta_*$ as in the lemma. A quick estimate of this integral then tells us that

$$|K_3(x, y)| \lesssim T^{-1}\lambda^{n-1} < \varepsilon\lambda^{n-1}$$

if we take $T > \varepsilon^{-1}$. Hence there exists some $\Lambda_{\text{large}} \gg 1$ and $\delta_{\text{large}} > 0$ for which $K_3(x, y) = o(\lambda^{n-1})$, whenever $d_g(x, y) < \delta_{\text{large}}$ and $\lambda \geq \Lambda_{\text{large}}$, just as desired.

3

Main Results

As before in the outline, each proof will have its own subsection:

3.1 Small Time Estimates

Proof of Theorem 1.1 (small time). Recall that we wish to show that $K_1 - J_\lambda$ is $o(\lambda^{n-1})$. More explicitly, we need to prove that for $\varepsilon > 0$ fixed, there exists some $\delta_* > 0$ and some finite $\Lambda_* > 1$ for which

$$|K_1(x,y) - J_\lambda(x,y)| \leq \varepsilon \lambda^{n-1} \tag{3.1.1}$$

for all $\lambda > \Lambda_*$ and all x, y with $d_g(x,y) < \delta_*$, where $K_1(x,y)$ is given by (2.0.2a) and $J_\lambda(x,y)$ is given by (2.0.1). In other words, we need that $|K_1(x,y) - J_\lambda(x,y)| = o(\lambda^{n-1})$.

Recall that to proceed we will need to make use of the Hadamard parametrix for $\cos(tP)$, the operator defined in (2.1.1). As outlined prior, we use Euler's identity to rewrite $K_1(x,y)$ as the sum of $K_{1,a}(x,y)$ and $K_{1,b}(x,y)$, given by (2.1.2a) and (2.1.2b) respectively. By the triangle inequality, we must have that

$$|K_1(x,y) - J_\lambda(x,y)| \leq |K_{1,a}(x,y) - J(x,y)| + |K_{1,b}(x,y)|,$$

and so we will address the asymptotic behavior of each term separately.

Focusing our attention first on (2.1.2b), by setting $\psi(t) = (2\pi)^{-1}\beta(t)\frac{\sin(t/2)}{t/2} \in C_0^\infty(\mathbb{R})$ we see that for any $N > n+1$ and $\lambda > 1$

$$
\begin{aligned}
|K_{1,b}(x,y)| &= \left|\widehat{\psi}\left(\lambda+\tfrac{1}{2}+P\right)(x,y)\right| \\
&= \left|\sum_{j=0}^{\infty}\widehat{\psi}(\lambda+\tfrac{1}{2}+\lambda_j)e_j(x)\overline{e_j(y)}\right| \\
&\lesssim \sum_{j=0}^{\infty}(1+\left|\tfrac{1}{2}+\lambda+\lambda_j\right|)^{-N}\left|e_j(x)\overline{e_j(y)}\right| \qquad (3.1.2) \\
&\lesssim \left(\sum_{j=0}^{\infty}(\lambda+\lambda_j)^{-N}|e_j(x)|^2\right)^{1/2}\left(\sum_{j=0}^{\infty}(\lambda+\lambda_j)^{-N}|e_j(y)|^2\right)^{1/2} \qquad (3.1.3) \\
&\lesssim \sum_{j=0}^{\infty}(\lambda+\lambda_j)^{-N}\lambda_j^{n-1} \approx \lambda^{-N+n} \qquad (3.1.4)
\end{aligned}
$$

where (3.1.2) follows from the fact that $\widehat{\psi} \in \mathcal{S}(\mathbb{R})$, (3.1.3) follows from Cauchy–Schwarz and an elementary inequality, and (3.1.4) follows from the the $L^\infty(M)$ Weyl estimates for eigenfunctions in [11] (cf. 3.2.2) and the integral comparison test. Hence $|K_{1,b}(x,y)| = O(\lambda^{-N})$ for any large enough N, and thus is negligible when considering the $o(\lambda^{n-1})$ bounds needed to prove our main result. Thus (3.1.1) will follow if we can show that the difference $K_{1,a} - J_\lambda$ also has the desired bounds.

As mentioned before, (2.1.2a) was introduced so that we could appeal to the Hadamard parametrix for $\cos(tP)$. To make this precise, if we define the following family of distributions,

$$E_\nu(t,x) = \lim_{\varepsilon\to 0^+} \nu!(2\pi)^{-n-1}\iint_{\mathbb{R}^{1+n}} e^{ix\cdot\xi+it\tau}(|\xi|^2-(\tau-i\varepsilon)^2)^{-\nu-1}d\xi d\tau, \quad \nu = 0,1,2,\ldots$$

with

$$E_0(t,x) = \frac{H(t)}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{\sin(t|\xi|)}{|\xi|}\,d\xi,$$

then [11] (cf. 3.1.5) tells us the following about the Hadamard parametrix:

Theorem 3.1 (Hadamard parametrix)**.** *Let (M,g) be a compact, n-dimensional Reimannian manifold. In local coordinates we have*

$$-\Delta_g = -\sum_{j,k=1}^{\infty}\frac{\partial}{\partial x_j}g^{jk}(x)\frac{\partial}{\partial x_k} + \sum_{k=1}^{n}a^k(x)\frac{\partial}{\partial x_k},$$

where

$$a^k(x) = -|g(x)|^{-1/2} \sum_{j=1}^{n} g^{jk}(x)\partial_j \left(|g(x)|^{1/2}\right).$$

We define

$$\rho(x) = \sum_{j,k=1}^{n} g_{jk}(x) a^k(x) x_j,$$

and require that $\alpha_0(x)$ solves the transport equation

$$\rho\alpha_0 = 2\langle x, \nabla_x \alpha_0 \rangle, \quad \alpha_0(0) = 1.$$

The there exists $\delta > 0$ so that if $N > n+3$ then, in local coordinates,

$$(\cos(tP)f)(x) = \int_{\mathbb{R}^n} K_N(t,x;y) f(y)\, dV_g(y) + \int_{\mathbb{R}^n} R_N(t,x;y) f(y)\, dV_g(y) \tag{3.1.5}$$

where

$$K_N(t,x;y) = \begin{cases} \partial_t \left(\sum_{\nu=0}^N \alpha_\nu(x,y) E_\nu(t, d_g(x,y))\right) & t \geq 0 \\ -\partial_t \left(\sum_{\nu=0}^N \alpha_\nu(x,y) E_\nu(-t, d_g(x,y))\right) & t < 0 \end{cases}, \tag{3.1.6}$$

with $\alpha_\nu \in C^\infty$, defined recursively by the formula

$$\alpha_\nu(x) = \alpha_0(x) \int_0^1 t^{\nu-1} \frac{\Delta_g \alpha_{\nu-1}(tx)}{\alpha_0(tx)}\, dt, \quad \nu = 1, 2, 3, \ldots$$

whereby $\alpha_\nu(x,y)$ is the pullback of α_ν along geodesic rays centered at y, with $\alpha_\nu(y,y) = 1$. Moreover, the remainder kernel $R_N \in C^{N-n-3}([-\delta,\delta] \times M \times M)$ satisfies

$$|\partial_{t,x,y}^\beta R_N(t,x;y)| \leq C|t|^{2N+2-n-|\beta|}, \quad \text{if } |\beta| \leq N - (n+2). \tag{3.1.7}$$

It's important to make one small comment before proceeding. We can, without loss of generality, assume that the $\delta = 1$ in the theorem above. Since if originally $\delta < 1$, we can simply rescale the metric until $\delta \geq 1$, or we could redefine our original cutoff $\beta(t)$ so that supp$\beta \subseteq [-\delta, \delta]$. In either case both solutions allow us to move forward as though $\delta = 1$.

Using this parametrix, we can choose any $N > n+3$ to decompose $K_{1,a}(x,y)$ into the sum of

$$K_{1,\nu} = \pi^{-1} \int_{-\infty}^{\infty} \beta(t) \frac{\sin(t/2)}{t/2} e^{-it(\lambda+1/2)} \partial_t \Big(\alpha_\nu \big(E_\nu(t, d_g(x,y)) - E_\nu(-t, d_g(x,y))\big)\Big)\, dt.$$

for $\nu = 0, 1, 2, \ldots, N$, and the remainder kernel,

$$K_{1,R} = \pi^{-1} \int_{-\infty}^{\infty} \beta(t)\, \frac{\sin(t/2)}{t/2}\, e^{-it(\lambda+1/2)} R_N(t, x; y)\, dt.$$

Moreover, since $E_\nu(t, x) = E_\nu(t, |x|)$ (i.e. E_ν is radial in x), by choosing the particular vector

$$x' = (0, \ldots, 0, d_g(x, y)) = d_g(x, y)\mathbb{1}_n \in \mathbb{R}^n,$$

we are able to use (3.1.6) to rewrite the leading term, $K_{1,0}(x, y)$, as

$$K_{1,0}(x, y) = \frac{\alpha_0(x, y)}{\pi(2\pi)^n} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \beta(t)\, \frac{\sin(t/2)}{t/2}\, e^{-it(\lambda+1/2)} e^{i d_g(x,y)\mathbb{1}_n \cdot \xi} \cos(t|\xi|)\, d\xi\, dt. \tag{3.1.8}$$

Using the triangle inequality, we see that

$$|K_{1,a}(x, y) - J_\lambda(x, y)| \le |K_{1,0}(x, y) - J_\lambda(x, y)| + \sum_{\nu=1}^{N} |K_{1,\nu}(x, y)| + |K_{1,R}(x, y)|.$$

Following the arguments at the end of section 3.3 in [11] (cf. 3.3.7) it follows that $|K_{1,\nu}(x, y)| \le C_\nu \lambda^{n-2\nu}$, for $\nu = 1, 2, \ldots$ and $\lambda \ge 1$, and (3.1.7) implies that $|K_{1,R}(x, y)| = O(1)$.

This only leaves us needing to prove that

$$|K_{1,0}(x, y) - J(x, y)| \le \varepsilon \lambda^{n-1}, \tag{3.1.9}$$

for λ sufficiently large, and x, y sufficiently close.

To get a handle on this new estimate we introduce a smooth cutoff function $\Psi(s) = C_0^\infty(\mathbb{R})$ with the following properties:

- $\Psi \ge 0$,
- $\Psi(s) \equiv 1$ in the interval $[1 - \delta', 1 + \delta']$,
- $\mathrm{supp}\Psi \subseteq [1 - 2\delta', 1 + 2\delta']$,

where δ' is chosen in the range $\lambda^{-1} < \delta' < 1/2$ for reasons that will soon become clear. Using our new cutoff function we can introduce a modified version of $K_{1,0}$, which is localized near $|\xi| = \lambda$, denoted by:

$$K_\Psi(x,y) := \frac{\alpha_0(x,y)}{\pi(2\pi)^n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\beta(t)\,\frac{\sin(t/2)}{t/2}\,e^{-it(\lambda+1/2)}e^{id_g(x,y)\mathbb{1}_n\cdot\xi}\cos(t|\xi|)\Psi(|\xi|/\lambda)\,d\xi\,dt. \quad (3.1.10)$$

Note that $\Psi(|\xi|/\lambda)$ is identically 1 for $|\xi| \approx \lambda$, and since it is compactly supported away from $|\xi| = 0$, then the $d\xi$ integral in K_Ψ has compact support, unlike in $K_{1,0}$. Specifically, $\Psi(|\xi|/\lambda) \equiv 0$ if $|\xi| \notin [(1-2\delta')\lambda, (1+2\delta')\lambda]$, where $(1-2\delta')\lambda > 0$ since $\delta' < 1/2$.

Introducing K_Ψ into (3.1.9) and using the triangle inequality gives us

$$|K_{1,0}(x,y) - J_\lambda(x,y)| \le |K_{1,0}(x,y) - K_\Psi(x,y)| + |K_\Psi(x,y) - J_\lambda(x,y)|,$$

meaning that our desired bounds will follow given that both $K_\Psi - J_\lambda$ and $K_{1,0} - K_\Psi$ share the same little oh bounds, or better.

We will handle the term $K_\Psi - J_\lambda$ in a moment, but first we focus on controlling the size of $K_{1,0} - K_\Psi$, which is equal to

$$\frac{\alpha_0(x,y)}{\pi(2\pi)^n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\beta(t)\,\frac{\sin(t/2)}{t/2}\,e^{-it(\lambda+1/2)}e^{id_g(x,y)\mathbb{1}_n\cdot\xi}\cos(t|\xi|)\Big(1-\Psi(|\xi|/\lambda)\Big)\,d\xi\,dt. \quad (3.1.11)$$

We start by using Euler's identity again, and integrating in t first. From this it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\beta(t)\frac{\sin(t/2)}{t/2}e^{-it(\lambda+1/2)}\cos(t|\xi|)\,dt &= \sum_{\pm}\int_{-\infty}^{\infty}\left(\beta(t)\frac{\sin(t/2)}{t/2}\frac{e^{-it/2}}{2}\right)e^{-it(\lambda\pm|\xi|)}\,dt\\ &= \sum_{\pm}\mathcal{F}_t\left[\beta(t)\frac{\sin(t/2)}{t/2}\frac{e^{-it/2}}{2}\right](\lambda\pm|\xi|), \qquad (3.1.12)\end{aligned}$$

where $\mathcal{F}_t$ denotes taking the Fourier transform in the argument t. Notice that since

$$\beta(t)\frac{\sin(t/2)}{t/2}\frac{e^{-it/2}}{2} \in C_0^\infty(\mathbb{R}),$$

then (3.1.12) will be a Schwartz class function, with asymptotic behavior $O\left((1+|\lambda\pm|\xi||)^{-N}\right)$, for any $N \ge 1$. Moreover, since $\lambda > 0$, we have $(1+|\lambda-\xi|)^{-N} > (1+|\lambda+\xi|)^{-N}$ for all $\xi \in \mathbb{R}^n$. Combining this observation with (3.1.12), we can conclude that for all $N \ge 1$, there exists $0 < C_N < \infty$ such that

$$\left|\int_{-\infty}^{\infty} \beta(t) \frac{\sin(t/2)}{t/2} e^{-it(\lambda+1/2)} \cos(t|\xi|)\, dt\right| \leq 2C_N(1+|\lambda-|\xi||)^{-N},$$

for all $\xi \in \mathbb{R}^n$ and $\lambda > 1$. Using this result in (3.1.11), and the fact that $1-\Psi(|\xi|/\lambda) \equiv 0$ when $(1-\delta')\lambda < |\xi| < (1+\delta')\lambda$, we apply polar coordinates $\xi = r\omega$, where $\omega \in \mathbb{S}^{n-1}$ to arrive at

$$\begin{aligned}
|K_{1,0}(x,y) - K_\Psi(x,y)| &\leq C'_N |\alpha_0(x,y)| \int_{\mathbb{R}^n} (1+|\lambda-|\xi||)^{-N} \Big(1-\Psi(|\xi|/\lambda)\Big)\, d\xi \\
&= C'_N \left|\mathbb{S}^{n-1}\right| |\alpha_0(x,y)| \int_0^\infty (1+|\lambda-r|)^{-N} \Big(1-\Psi(r/\lambda)\Big)\, r^{n-1}\, dr \\
&= C''_N |\alpha_0(x,y)| \int_{\substack{r<(1-\delta')\lambda \\ r>(1+\delta')\lambda}} (1+|\lambda-r|)^{-N} \Big(1-\Psi(r/\lambda)\Big)\, r^{n-1}\, dr \\
&\leq 2C''_N |\alpha_0(x,y)| \int_{(1+\delta')\lambda}^\infty (1+r-\lambda)^{-N}\, r^{n-1}\, dr \\
&\leq C'''_N |\alpha_0(x,y)| \int_{(1+\delta')\lambda}^\infty (r-\lambda)^{-N}\, r^{n-1}\, dr \\
&= C'''_N |\alpha_0(x,y)| \int_{\delta'\lambda}^\infty \rho^{-N} (\rho+\lambda)^{n-1}\, d\rho \\
&\leq C_{N,\delta'}\, |\alpha_0(x,y)|\, \lambda^{-N+n},
\end{aligned}$$

assuming that $N > n$. Moreover, since $\alpha_0(x,x) = 1$, we must have that if $x, y \in M$ are sufficiently close, with a fixed N sufficiently large, then we have

$$\sup_{\substack{x,y\in M \\ d_g(x,y)<\delta_*}} |K_{1,0}(x,y) - K_\Psi(x,y)| \lesssim \lambda^{-N+n},$$

if $\Lambda_* \gg 1$ is large enough, and $\delta_* > 0$ small enough, say so that $|\alpha_0(x,y)| > 1/2$. Hence this error term is negligible when considering our desired $o(\lambda^{n-1})$ bounds. Therefore (3.1.9) will follow if we can show that

$$|K_\Psi(x,y) - J(x,y)| \leq \varepsilon\lambda^{n-1}. \tag{3.1.13}$$

As is mentioned before in the outline, the Fourier transform of the spherical Lebesgue measure is given by (2.1.4). By making the specific choice of $z' = \lambda d_g(x,y)\mathbb{1}_n$ as we did in (3.1.8), we see that $|z'| = \lambda d_g(x,y)$ and thus (2.0.1) can be rewritten as

$$J_\lambda(x,y) = \frac{\lambda^{n-1}}{(2\pi)^n}\int_{\mathbb{S}^{n-1}} e^{iz'\cdot\omega}\,d\sigma(\omega) = \frac{\lambda^{n-1}}{(2\pi)^n}\int_{\mathbb{S}^{n-1}} e^{i\lambda d_g(x,y)\omega_n}\,d\sigma(\omega). \tag{3.1.14}$$

In order to proceed with (3.1.13) we introduce $I_\lambda(x,y)$, which will agree with $K_\Psi(x,y)$ except for the fact that it is missing the term $\beta(t)$:

$$\begin{aligned} I_\lambda(x,y) &:= \frac{\alpha_0(x,y)}{\pi(2\pi)^n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \frac{\sin(t/2)}{t/2}\, e^{-it(\lambda+1/2)} e^{id_g(x,y)\mathbb{1}_n\cdot\xi}\cos(t|\xi|)\Psi(|\xi|/\lambda)\,d\xi\,dt \\ &= \frac{\alpha_0(x,y)}{\pi(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{-\infty}^{\infty}\frac{\sin(t/2)}{t/2}\cos(t|\xi|)e^{-it(\lambda+1/2)}\,dt\right)\Psi(|\xi|/\lambda)e^{id_g(x,y)\mathbb{1}_n\cdot\xi}\,d\xi \end{aligned} \tag{3.1.15}$$

Applying the triangle inequality one last time, we arrive at

$$|K_\Psi(x,y) - J_\lambda(x,y)| \le |K_\Psi(x,y) - I_\lambda(x,y)| + |I_\lambda(x,y) - J_\lambda(x,y)|,$$

meaning that (3.1.13) will follow if we can prove similar bounds for each of the expressions above. Before we try tackling these two final inequalities, however, it's worth simplifying the expression for $I_\lambda(x,y)$ first. Looking at first to the t-integral in (3.1.15), we see that

$$\int_{-\infty}^{\infty}\frac{\sin(t/2)}{t/2}\cos(t|\xi|)e^{-it(\lambda+1/2)}\,dt = \mathcal{F}_t\left[\frac{\sin(t/2)}{t/2}\cos(t|\xi|)\right](\lambda+1/2).$$

Since $\beta(t)$ is no longer present in this integral, the function is no longer localized near $t = 0$, which allows us to compute the Fourier transform directly. Looking back at (1.2.13), and applying elementary properties of Fourier transforms, we get

$$\begin{aligned} \mathcal{F}_t\left[\frac{\sin(t/2)}{t/2}\cos(t|\xi|)\right] &= (2\pi)^{-1}\left(\mathcal{F}_t\left[\frac{\sin(t/2)}{t/2}\right] * \mathcal{F}_t\left[\cos(t|\xi|)\right]\right) \\ &= (2\pi)^{-1}\Big(2\pi\chi_{-1/2}(\tau) * \pi\left(\delta(\tau-|\xi|)+\delta(\tau+|\xi|)\right)\Big) \\ &= \pi\Big(\chi_{-1/2}(\tau-|\xi|)+\chi_{-1/2}(\tau+|\xi|)\Big) \end{aligned}$$

This means, that after substituting back into the equation above, we have

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{\sin(t/2)}{t/2}\cos(t|\xi|)e^{-it(\lambda+1/2)}\,dt &= \pi\sum_{\pm}\chi_{-1/2}\left(\lambda+\tfrac{1}{2}\pm|\xi|\right) \\ &= \pi\chi_{-1/2}\left(\lambda+\tfrac{1}{2}-|\xi|\right) \\ &= \pi\chi_\lambda(|\xi|) \end{aligned}$$

since $\lambda \gg 1$, and so $\chi_{-1/2}\left(\lambda + \frac{1}{2} + |\xi|\right) = 0$. This gives us the following alternate expression for (3.1.15):

$$I_\lambda(x,y) = \frac{\alpha_0(x,y)}{(2\pi)^n} \int_{\mathbb{R}^n} \chi_\lambda\left(|\xi|\right) \Psi(|\xi|/\lambda) e^{id_g(x,y)\mathbb{1}_n \cdot \xi} \, d\xi.$$

However, recall that Ψ was constructed so that $\Psi(|\xi|/\lambda) \equiv 1$ if $|\xi| \in [(1-\delta')\lambda, (1+\delta')\lambda]$, with $\lambda^{-1} < \delta'$. This means that $\Psi(|\xi|/\lambda) \equiv 1$ on $\mathrm{supp}_\xi \chi_\lambda(|\xi|)$, since

$$(1-\delta')\lambda < \lambda \leq |\xi| \leq \lambda + 1 < (1+\delta')\lambda,$$

and so our final expression for $I_\lambda(x,y)$ simplifies to

$$\begin{aligned} I_\lambda(x,y) &= (2\pi)^{-n} \alpha_0(x,y) \int_{\mathbb{R}^n} \chi_\lambda(|\xi|) e^{id_g(x,y)\mathbb{1}_n \cdot \xi} \, d\xi \\ &= (2\pi)^{-n} \alpha_0(x,y) \int_{|\xi| \in [\lambda, \lambda+1]} e^{id_g(x,y)\mathbb{1}_n \cdot \xi} \, d\xi. \end{aligned} \tag{3.1.16}$$

In particular this means that — aside from the terms $(2\pi)^{-n}\alpha_0(x,y)$ — $I_\lambda(x,y)$ is the Fourier transform of the characteristic function of the annulus with inner radius λ and thickness 1, evaluated at some vector with magnitude $d_g(x,y)$. Compare this with $J_\lambda(x,y)$, which involves the Fourier transform of the spherical measure on a sphere of radius λ, also evaluated at some vector with magnitude $d_g(x,y)$. As we will show momentarily, these two integrals are closely related with their difference $|I_\lambda(x,y) - J_\lambda(x,y)|$ being of size $o(\lambda)$ as desired. To show this we will need the following lemma:

Lemma 3.2. *Suppose that $z \in \mathbb{R}^n$, with $n \geq 2$. Then there exists $C > 0$ and $\Lambda > 0$ such that*

$$\left| \int_{|\xi| \in [\lambda, \lambda+1]} e^{iz\cdot\xi} \, d\xi - \lambda^{n-1} \int_{\mathbb{S}^{n-1}} e^{i\lambda(z\cdot\omega)} d\sigma(\omega) \right| \leq C\left(|z|\lambda^{n-1} + \lambda^{n-2}\right), \tag{3.1.17}$$

for all $\lambda > \Lambda$.

Proof of lemma. We begin by re-expressing the first term using polar coordinates $\xi = r\omega$, where $r > 0$ and $\omega \in \mathbb{S}^{n-1}$:

$$\begin{aligned} \int_{|\xi| \in [\lambda, \lambda+1]} e^{iz\cdot\xi} \, d\xi &= \int_{\mathbb{S}^{n-1}} \int_\lambda^{\lambda+1} r^{n-1} e^{i(z\cdot r\omega)} \, dr d\omega \\ &= \int_{\mathbb{S}^{n-1}} \int_\lambda^{\lambda+1} f(r,z,\omega) \, dr d\omega, \end{aligned}$$

where $f(r,z,\omega) = r^{n-1}e^{ir(z\cdot\omega)}$. We then rewrite the inner integral as follows:

$$\begin{aligned}
\int_\lambda^{\lambda+1} f(r,z,\omega)\,dr &= f(\lambda,z,\omega) - f(\lambda,z,\omega) + \int_\lambda^{\lambda+1} f(r,z,\omega)\,dr \\
&= f(\lambda,z,\omega) + \int_\lambda^{\lambda+1} \left(f(r,z,\omega) - f(\lambda,z,\omega)\right)\,dr \\
&= f(\lambda,z,\omega) + \int_\lambda^{\lambda+1}\int_\lambda^r \frac{\partial f}{\partial s}(s,x,\omega)\,dsdr,
\end{aligned}$$

where

$$\frac{\partial f}{\partial r}(r,z,\omega) = ir^{n-1}(z\cdot\omega)e^{ir(z\cdot\omega)} + (n-1)r^{n-2}e^{ir(z\cdot\omega)}.$$

Note that $f(\lambda,z,\omega) = \lambda^{n-1}e^{i\lambda(z\cdot\omega)}$, meaning that

$$\begin{aligned}
\left|\int_{|\xi|\in[\lambda,\lambda+1]} e^{iz\cdot\xi}\,d\xi - \int_{\mathbb{S}^{n-1}} f(\lambda,z,\omega)\,d\sigma(\omega)\right| &\le \int_\lambda^{\lambda+1}\int_\lambda^r\int_{\mathbb{S}^{n-1}}\left|\frac{\partial f}{\partial s}\right| d\sigma(\omega)dsdr \\
&\le \int_\lambda^{\lambda+1}\int_\lambda^{\lambda+1}\int_{\mathbb{S}^{n-1}}\left|\frac{\partial f}{\partial s}\right| d\sigma(\omega)dsdr \\
&= \int_\lambda^{\lambda+1}\int_{\mathbb{S}^{n-1}}\left|\frac{\partial f}{\partial s}\right| d\sigma(\omega)ds.
\end{aligned}$$

Whereby

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}}\left|\frac{\partial f}{\partial s}\right| d\sigma(\omega) &\le \int_{\mathbb{S}^{n-1}} s^{n-1}\,|z\cdot\omega|\left|e^{ir(z\cdot\omega)}\right| + (n-1)s^{n-2}\left|e^{ir(z\cdot\omega)}\right| d\sigma(\omega) \\
&\le s^{n-1}\int_{\mathbb{S}^{n-1}} |z||\omega|d\sigma(\omega) + (n-1)s^{n-2}\int_{\mathbb{S}^{n-1}} d\sigma(\omega) \\
&= \left|\mathbb{S}^{n-1}\right|\left(|z|s^{n-1} + (n-1)s^{n-2}\right)
\end{aligned}$$

Substituting this expression back into the ds integral, we finally arrive at

$$C_n|z|\int_\lambda^{\lambda+1} s^{n-1}\,ds + (n-1)C_n\int_\lambda^{\lambda+1} s^{n-2}\,ds = C_n\big(|z|p_{n-1}(\lambda) + p_{n-2}(\lambda)\big),$$

where $p_k(\lambda)$ is a polynomial, in λ, of degree k. Hence there exist finite values $C_n', \Lambda > 1$ such that

$$\left|\int_{|\xi|\in[\lambda,\lambda+1]} e^{iz\cdot\xi}\,d\xi - \lambda^{n-1}\int_{\mathbb{S}^{n-1}} e^{i\lambda(z\cdot\omega)}d\sigma(\omega)\right| \le C_n'(|z|\lambda^{n-1} + \lambda^{n-2})$$

for all $\lambda > \Lambda$, where C_n' depends only on n. Thus we have proven the desired result.

$\square$

Looking at the expressions for $I_\lambda(x,y)$ and $J_\lambda(x,y)$ in (3.1.16) and (3.1.14), we immediately see that we can apply (3.1.17) from our lemma to get

$$\begin{aligned}|I_\lambda(x,y) - J_\lambda(x,y)| &= (2\pi)^{-n}\left|\alpha_0(x,y)\int_{|\xi|\in[\lambda,\lambda+1]} e^{id_g(x,y)\mathbb{1}_n\cdot\xi}\,d\xi - \lambda^{n-1}\int_{\mathbb{S}^{n-1}} e^{i\lambda(d_g(x,y)\mathbb{1}_n\cdot\omega)}d\sigma(\omega)\right| \\ &\lesssim \left|\int_{|\xi|\in[\lambda,\lambda+1]} e^{id_g(x,y)\mathbb{1}_n\cdot\xi}\,d\xi - \lambda^{n-1}\int_{\mathbb{S}^{n-1}} e^{i\lambda(d_g(x,y)\mathbb{1}_n\cdot\omega)}d\sigma(\omega)\right| \\ &\lesssim d_g(x,y)\lambda^{n-1} + \lambda^{n-2}, \qquad (3.1.18)\end{aligned}$$

since α_0 is smooth with $\alpha_0(y,y)=1$, as mentioned earlier, and by choosing $z' = d_g(x,y)\mathbb{1}_n$. Hence, by requiring that $d_g(x,y) < \varepsilon$, we get $|I_\lambda(x,y) - J_\lambda(x,y)| \lesssim \varepsilon\lambda^{n-1} + \lambda^{n-2}$ for $\lambda \gg 1$ sufficiently large, as desired. In other words, $|I_\lambda(x,y) - J_\lambda(x,y)| = o(\lambda^{n-1})$ when $d_g(x,y) < \delta_*$, where $\delta_* = \varepsilon$ might be smaller than before. This only leaves one final comparison: $|K_\Psi(x,y) - I_\lambda(x,y)|$.

Looking back at (3.1.15), we see that $K_\Psi - I_\lambda$ is equal to

$$\frac{\alpha_0(x,y)}{\pi(2\pi)^n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}(1-\beta(t))\frac{\sin(t/2)}{t/2}e^{-it(\lambda+1/2)}e^{id_g(x,y)\mathbb{1}_n\cdot\xi}\Psi(|\xi|/\lambda)\cos(t|\xi|)\,d\xi dt. \qquad (3.1.19)$$

Applying Euler's identity and working in polar coordinates again, we collect the radial terms and take the integral in r first:

$$\begin{aligned}\int_0^\infty e^{id_g(x,y)r\omega_n}\Psi(r/\lambda)\cos(tr)r^{n-1}\,dr &= \frac{1}{2}\sum_{\pm}\int_0^\infty e^{id_g(x,y)r\omega_n}\Psi(r/\lambda)e^{\pm itr}r^{n-1}\,dr \\ &= \frac{1}{2}\sum_{\pm}\int_0^\infty \Psi(r/\lambda)e^{ir(\pm t + d_g(x,y)\omega_n)}r^{n-1}\,dr. \qquad (3.1.20)\end{aligned}$$

Borrowing a standard technique from stationary phase, we note that for all $N \in \mathbb{N}$

$$\frac{d^N}{dr^N}(e^{ir(\pm t + d_g(x,y)\omega_n)}) = (i(\pm t + d_g(x,y)\omega_n))^N\, e^{ir(\pm t + d_g(x,y)\omega_n)}.$$

Moreover, since $\beta(t) \equiv 1$ when $|t| < 1/2$, and (3.1.19) involves $1-\beta(t)$, then the integration in t will only include values of $|t| > 1/2$. Hence t is bounded from below. We also have

$$|\pm t + d_g(x,y)\omega_n| \geq \left||t| - d_g(x,y)|\omega_n|\right|,$$

by the reverse triangle inequality. If we supposed that $d_g(x,y) < 1/2$, then since $|\omega_n| \leq 1$ and $|t| > 1/2$, we must have

$$|t| > 1/2 > d_g(x,y) \geq d_g(x,y)|\omega_n|$$

$$\implies |t| - d_g(x,y)|\omega_n| > 0,$$

and hence $\pm t + d_g(x,y)\omega_n$ must be bounded from below as well. This allows us to define the operator

$$L := \frac{1}{i(\pm t + d_g(x,y)\omega_n)} \frac{d}{dr},$$

for which

$$L^N[e^{ir(\pm t + d_g(x,y)\omega_n)}] = e^{ir(\pm t + d_g(x,y)\omega_n)}, \quad \text{for all } N \in \mathbb{N}.$$

Using the fact that $\Psi(r/\lambda)$ has compact support away from zero, we are able to use integration by parts to manipulate (3.1.20) as follows:

$$\begin{aligned}\left|\int_0^\infty L^N[e^{ir(\pm t + d_g(x,y)\omega_n)}]\Psi(r/\lambda) r^{n-1}\,dr\right| &= \left|\int_0^\infty e^{ir(\pm t + d_g(x,y)\omega_n)} L^N[\Psi(r/\lambda) r^{n-1}]\,dr\right| \\ &\leq \frac{1}{|\pm t + d_g(x,y)\omega_n|^N} \int_0^\infty \left|\frac{d^N}{dr^N}\left(\Psi(r/\lambda) r^{n-1}\right)\right| dr \\ &\lesssim \frac{1}{|t|^N} \int_{\lambda(1-2\delta')}^{\lambda(1+2\delta')} \left|\frac{d^N}{dr^N}\left(\Psi(r/\lambda) r^{n-1}\right)\right| dr, \end{aligned} \tag{3.1.21}$$

since $\operatorname{supp}\Psi(t) = [1-2\delta', 1+2\delta']$. Moreover, by the general Leibniz rule we have

$$\begin{aligned}\frac{d^N}{dr^N}\left(\Psi(r/\lambda) r^{n-1}\right) &= \sum_{k=0}^{N} \binom{N}{k} \frac{d^{N-k}}{dr^{N-k}}\left(\Psi(r/\lambda)\right) \frac{d^k}{dr^k}\left(r^{n-1}\right) \\ &= \sum_{k=0}^{n-1} \binom{N}{k} \left(\lambda^{-N+k}\Psi^{(N-k)}(r/\lambda)\right)\left(\frac{(n-1)!}{(n-1-k)!} r^{n-1-k}\right),\end{aligned}$$

meaning

$$\begin{aligned}\left|\frac{d^N}{dr^N}\left(\Psi(r/\lambda) r^{n-1}\right)\right| &\leq \sum_{k=0}^{n-1} C_{N,n,k} ||\Psi^{(N-k)}(r/\lambda)||_\infty\, \lambda^{-N+k}\, r^{n-1-k} \\ &\leq C_{N,n} \sum_{k=0}^{n-1} \lambda^{-N+k}\, r^{n-1-k}.\end{aligned}$$

Substituting this back into (3.1.21) we have

$$\begin{aligned}\frac{C_{N,n}}{|t|^N}\sum_{k=0}^{n-1}\lambda^{-N+k}\int_{\lambda(1-2\delta')}^{\lambda(1+2\delta')} r^{n-1-k}\,dr &\lesssim \frac{1}{|t|^N}\sum_{k=0}^{\infty}\lambda^{-N+k}(\delta'\lambda^{n-k})\\ &\lesssim \frac{1}{|t|^N}\lambda^{-N+n}.\end{aligned}$$

Looking back now to (3.1.20) and (3.1.19), we finally have

$$\begin{aligned}|K_\Psi(x,y)-I_\lambda(x,y)| &\lesssim |\alpha_0(x,y)|\lambda^{-N+n}\int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}\left|\frac{1-\beta(t)}{|t|^N}\frac{\sin(t/2)}{t/2}e^{-it(\lambda+1/2)}\right|d\sigma(\omega)dt\\ &\leq |\alpha_0(x,y)|\lambda^{-N+n}\left|\mathbb{S}^{n-1}\right|\int_{-\infty}^{\infty}\frac{1-\beta(t)}{|t|^N}\left|\frac{\sin(t/2)}{t/2}\right|dt\\ &\lesssim |\alpha_0(x,y)|\lambda^{-N+n},\end{aligned}$$

so long as $N>1$, seeing as $1-\beta(1)\equiv 0$ on a neighborhood of $t=0$, and hence the singularity at $t=0$ is not present in the integral. Thus, since we have already addressed the fact that x and y are close enough for $\alpha_0(x,y)\approx 1$, we get

$$|K_\Psi(x,y)-I_\lambda(x,y)| = O(\lambda^{-N+n}),$$

for any $N\in\mathbb{N}$, and hence is negligible when considering our main result. Therefore, having addressed every part individually, we are finally able to say that given any $\varepsilon>0$, there exists a $\delta_*>0$ (depending only on ε and the manifold, M) for which

$$\sup_{\substack{x,y\in M\\ d_g(x,y)<\delta_*}}|K_1(x,y)-J_\lambda(x,y)|\lesssim \varepsilon\lambda^{n-1}.$$

□

3.2 Medium Time Estimates

Proof of Theorem 1.1 (medium time). We now focus on proving the $o(\lambda^{n-1})$ bounds for K_2. As mentioned earlier, the proofs follow from techniques similar to those found in section 5.4 of [11].

With $\varepsilon>0$ fixed, we wish to show that for λ large enough, and $d_g(x,y)$ small enough, we have $|K_2(x,y)|\lesssim\varepsilon\lambda^{n-1}$, or in other words $K_2(x,y)=o(\lambda^{n-1})$. For the techniques that follow, it will be beneficial to fix some $x_0 \in \Omega$ in our local coordinate patch (which by our hypothesis must be a non self-focal point) and work with x, y in a neighborhood of x_0. Recall that $\mathcal{L}_x$ is the set of cotangent vectors which produce a geodesic loop through x. Using this idea, we define the length functional $L(x,\xi) : S^*M \to (0,+\infty]$, where $L(x,\xi) = |t|$ is defined to be the smallest time $t \neq 0$ for which $\Phi_t(x,\xi) = (x,\eta)$ for some $\eta \in S_x^*M$. If no such t exits, then we set $L(x,\xi) = +\infty$. As mentioned in the outline, this allows us the following definitions:

$$\mathcal{L}_x = \{\xi \in S_x^*M : L(x,\xi) < \infty\},$$

$$\mathcal{L}_{x,T} := \{\xi \in S_x^*M : L(x,\xi) \leq T\}.$$

Note that our geodesics are naturally parametrized with respect to arclength since they are defined by the Hamiltonian flow of the principal symbol of the Laplace-Beltrami operator. This means that the geodesic distance traversed is the same as the amount of time t that has passed. Furthermore, since we are working locally on a small neighborhood of x_0 in T^*M, we may identify $S_{x_0}^*M$ with $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ in the natural way, meaning that $(x,\xi) \in T^*M \simeq \tilde{\Omega} \times \mathbb{S}^{n-1}$, where $\tilde{\Omega} \subseteq \mathbb{R}^n$ is a coordinate patch of $x_0 \in M$. Moreover, this means that $L(x,\xi)$ is homogeneous of degree zero in ξ, and is lower semicontinuous [11]:

$$L(x, r\xi) = L(x,\xi), \quad \text{for all } r > 0,$$

$$\liminf_{(x,\xi)\to(x_0,\xi_0)} L(x,\xi) = L(x_0,\xi_0).$$

We turn our attention now to $\mathcal{L}_{x_0,T}$. We tend to think of these as the "bad directions" at x_0, since this set contains all of the cotangent vectors which produce geodesics that loop back in time T or less. Conversely, the complement, $\mathcal{L}_{x_0,T}^{\complement} = \mathbb{S}^{n-1} \setminus \mathcal{L}_{x_0,T}$, can be thought of as the "good directions", since any cotangent vector in this set will either never loop back to x_0, or will loopback only after some time greater than T.

The fact that $L(x,\xi)$ is lower semi-continuous means that the lower level sets,

$$L_T^- = \{(x,\xi) \in \Omega \times \mathbb{S}^{n-1} : L(x,\xi) \leq T\},$$

are closed for all $T \in \mathbb{R}$ [17]. Specifically this means that

$$\mathcal{L}_{x_0,T} \simeq \{(x,\xi) \in T^*M : x = x_0, L(x_0,\xi) \leq T\} \subset L_T^-$$

is closed, and has measure zero since $\mathcal{L}_{x_0,T} \subseteq \mathcal{L}_{x_0}$ with $|\mathcal{L}_{x_0}| = 0$; again, measure here is understood as the Lebesgue measure on $\mathbb{S}^{n-1}$. A key component of our proof here will be that since lower level sets of L are closed, then their complements must be open. In particular this means that if x is close to x_0, and if ξ is close to some "good direction" $\xi_0 \in S^*_{x_0}M$, then $L(x,\xi) > T$ since $L(x_0,\xi_0) > T$. This will allow us to make precise the idea that if a geodesic from x_0, in the direction ξ_0, does not return in time T or less, then there must be a region around (x_0,ξ_0) where none of the points return to this region in time T or less either.

As outlined above, since $\mathcal{L}_{x_0,T}$ has measure zero, we are able to find open subsets $\mathcal{V}_0, \mathcal{V}_1 \subseteq S^*_{x_0}M$ such that

$$\mathcal{L}_{x_0,T} \subset \mathcal{V}_0 \Subset \mathcal{V}_1,$$

with $\mathcal{V}_1$ of arbitrarily small measure; specifically we will require that $|\mathcal{V}_1| < cT^{-1}$, where $c > 0$ is to be chosen later. By construction we have,

$$\mathcal{V}_1^{\complement} \Subset \mathcal{V}_0^{\complement} \subset \mathcal{L}_{x_0,T}^{\complement},$$

and so for all $\xi \in \mathcal{V}_0^{\complement}$ we must have $L(x_0,\xi) > T$. Moreover, as mentioned above, since $L(x,\xi)$ is lower semicontinuous then the compliment of L_T^-,

$$\left(L_T^-\right)^{\complement} = \left\{(x,\xi) \in \tilde{\Omega} \times \mathbb{S}^{n-1} : L(x,\xi) > T\right\},$$

will be open. This means that for every $\xi \in \mathcal{V}_0^{\complement}$, $(x_0,\xi) \in \left(L_T^-\right)^{\complement}$, and so there must be an open neighborhood of (x_0,ξ), call it $\mathcal{W}_\xi \subset \left(L_T^-\right)^{\complement}$, for which $L(y,\eta) > T$ for all $(y,\eta) \in \mathcal{W}_\xi$. For convenience, we will refer to $\mathcal{U}_\xi$ and $\mathcal{V}_\xi$ as the projections of $\mathcal{W}_\xi$ onto $\tilde{\Omega}$ and $\mathbb{S}^{n-1}$ respectively: $\mathcal{W}_\xi = \mathcal{U}_\xi \times \mathcal{V}_\xi$, with $x_0 \in \mathcal{U}_\xi$ and $\xi \in \mathcal{V}_\xi$.

Hence if we consider the collection $\{\mathcal{W}_\xi\}_{\xi \in \mathcal{V}_0^{\complement}}$, then $\{\mathcal{V}_\xi\}_{\xi \in \mathcal{V}_0^{\complement}}$ must be an open cover of $\mathcal{V}_0^{\complement}$, which is a compact set. Therefore, there must be a finite subset $\{\mathcal{W}_{\xi_1}, \ldots, \mathcal{W}_{\xi_N}\}$ for which $\{\mathcal{V}_{\xi_1}, \ldots, \mathcal{V}_{\xi_N}\}$ still forms an open cover of $\mathcal{V}_0^{\complement}$.

Consider the open sets $\mathcal{U} = \bigcap_{j=1}^{N} \mathcal{U}_{\xi_j}$ and $\mathcal{V} = \bigcup_{j=1}^{N} \mathcal{V}_{\xi_j}$. Then $\mathcal{U}$ is an open neighborhood of x_0, while $\mathcal{V}$ is an open cover of $\mathcal{V}_0^{\complement}$, and $L(y,\eta) > T$ for all $y \in \mathcal{U}$ and all $\eta \in \mathcal{V}$. Finally, we pick $0 < \delta_* < \delta_*'$ smaller than the injectivity radius of M, so that $B_{\delta_*}(x_0) \Subset B_{\delta_*'}(x_0) \Subset \mathcal{U}$ and

$$d_g(x_0, y) \leq \delta_*' \implies L(y,\eta) > T \quad \text{for all } \eta \in \mathcal{V}_0^{\complement}. \tag{3.2.1}$$

We now construct a pair of cutoff functions — one directional, one spacial — to help us control the behavior of $K_2(x,y)$ over the open ball $B_{\delta_*'}(x_0)$. First, by Urysohn's lemma, we can construct $\sigma \in C^\infty(\mathbb{S}^{n-1})$ with the following properties:

- $0 \leq \sigma(\eta) \leq 1$ for all η,
- $\sigma(\eta) = 1$ for all $\eta \in \mathcal{V}_1^{\complement}$,
- $\sigma(\eta) = 0$ for all $\eta \in \overline{\mathcal{V}_0}$.

That is σ is a smooth, bounded, and non-negative function, supported away from $\mathcal{V}_0$, and identically 1 outside of $\mathcal{V}_1$. In particular, this means $\mathcal{L}_{x_0,T}$ is contained outside of the support of $\sigma(\eta)$, so the function ignores the set of bad directions entirely, transitioning to the identity quickly as you move away from the set.

We create another non-negative cutoff function, $\psi \in C_0^\infty(\Omega)$, on a neighborhood of x_0. We require it to have the following properties:

- $\operatorname{supp}\psi \subseteq B_{\delta_*'}(x_0)$,
- $\psi(y) = 1$ for all $y \in B_{\delta_*}(x_0)$.

We now combine both $\sigma(\eta)$ and $\psi(y)$ into one function defined on the larger cotangent bundle $T^*M\backslash 0$ as follows:

$$B(y,\eta) = \psi(y)\sigma(\eta/|\eta|) \quad \text{for } \eta \in \mathbb{R}^n \setminus 0. \tag{3.2.2}$$

Especially note that

$$(y,\eta) \in \operatorname{supp} B \implies L(y,\eta) > T, \tag{3.2.3}$$

since $y \in B_{\delta_*'}(x_0)$ and, by (3.2.1), $\eta \notin \overline{\mathcal{V}_0} \iff \eta \in \left(\overline{\mathcal{V}_0}\right)^{\complement} \subset \mathcal{V}_0^{\complement}$. Hence we will tend to refer to $B(y,\eta)$ as our "good" function.

Similarly, we define the complementary function $b(y,\eta)$ in the following way:

$$b(y,\eta) = \psi(y)\,(1-\sigma(\eta/|\eta|)) \quad \text{for } \eta \in \mathbb{R}^n \setminus 0. \tag{3.2.4}$$

Seeing as the η-support is the complement of our "good" function, we will refer to $b(y,\eta)$ as our "bad" function. We now prove an important lemma about geodesic flow:

Lemma 3.3. *Suppose $T \gg 1$ is fixed, and $B(y,\eta)$ as above. Then there exists $\delta_T > 0$ so that if $(x,\xi) \in \mathrm{supp}B$, $(y,\eta) \in \mathrm{supp}B$, and $d_g(x,y) < \delta_T$ then*

$$d_g(\Phi_t(x,\xi),y), d_g(\Phi_t(y,\eta),x) > 0 \quad \textit{for all } 1 \leq |t| \leq T, \tag{3.2.5}$$

where $\Phi_t(x,\xi) \in M$ is the (homogeneous) geodesic flow of x, in the direction ξ, after time t.

Proof of lemma. Suppose T is fixed, and consider the set $\mathrm{supp}B \times [1,T]$. Without loss of generality, we may assume that for any $(x,\xi) \in \mathrm{supp}B$ we have $\xi \in S^*_x M$, since B is homogeneous of degree 0 in ξ. This means that under our assumptions both $\mathrm{supp}B$ and $[1,T]$ are compact, and hence this Cartesian product is compact as well.

We now consider the function

$$\begin{aligned} \phi : \mathrm{supp}B \times [1,T] &\to \mathbb{R} \\ (x,\xi,t) &\mapsto d_g(\Phi_t(x,\xi),x). \end{aligned}$$

Since this is a continuous function on a compact domain, it must attain a minimum

$$\phi(x,\xi,t) \geq d_g(\Phi_{t'}(x',\xi'),x') = \delta' > 0,$$

which will be positive since $(x',\xi') \in \mathrm{supp}B$ means that x' will not flow back to itself at any time $|t| \leq T$. In other words, there is some triple $(x',\xi',t') \in \mathrm{supp}B \times [1,T]$ for which no other starting point $x \in B_{\delta'_*}(x_0)$, no "good" direction $\xi \in \left(\overline{\mathcal{V}_0}\right)^{\complement}$, nor any time $1 \leq |t| \leq T$ will ever cause x to return closer to itself than x' returns to *itself*. Specifically this means that δ' depends on T, but is independent of $\mathrm{supp}B$.

We now set $\delta_T = \delta'/2 < 1$, and pick arbitrary $(x,\xi),(y,\eta) \in \mathrm{supp}B$ with $d_g(x,y) < \delta_T$. Obviously there is a geodesic of length less than δ_T that connects x and y, but since we have that $d_g(x,y) < \delta'/2$ and $d_g(\Phi_t(x,\xi),x) \geq \delta'$ for all $1 \leq |t| \leq T$, then the triangle inequality gives us

$$\begin{aligned} d_g(x,\Phi_t(x,\xi)) &\leq d_g(x,y) + d_g(y,\Phi_t(x,\xi)) \\ \implies d_g(x,\Phi_t(x,\xi)) - d_g(x,y) &\leq d_g(y,\Phi_t(x,\xi)) \\ \implies \delta'/2 &< d_g(y,\Phi_t(x,\xi)), \end{aligned}$$

and so $d_g(\Phi_t(x,\xi),y) > 0$ for $1 < |t| < T$. Conversely, if we exchange x with y, and ξ with η, then we also have $d_g(\Phi_t(y,\eta),x) > 0$ for all $1 < |t| < T$. In other words, if x and y are δ_T-close, then geodesic flow from either point will miss the other for times $1 < |t| < T$. □

This lemma establishes the existence of an open "ball of no return" centered around x_0. Without loss of generality, we may take this ball to be our original $B_{\delta_*}(x_0)$, since if δ_T from the lemma would be smaller than our original δ_*, we can simply choose $\delta_* = \delta_T$.

We now pull back the operators $B(y,D)$ and $b(y,D)$ — defined in local coordinates with symbols $B(y,\eta)$ and $b(y,\eta)$ respectively — to M, giving us the following classical pseudodifferential operators of order 0:

$$B(x,D), b(x,D) \in \Psi^0_{\text{cl}}(M).$$

Specifically this means that in every local coordinate system we have

$$B(y,\eta) \sim \sum_{j=0}^{\infty} B_{0-j}(y,\eta), \quad b(y,\eta) \sim \sum_{j=0}^{\infty} b_{0-j}(y,\eta), \tag{3.2.6}$$

where $B_{0-j}, b_{0-j} \in S^{0-j}$ are symbols of order $0-j$ that are homogeneous of degree $0-j$ in η. Furthermore, we refer to the top-order terms, B_0 and b_0, as the principal symbol of the pseudodifferential operators. Notice that this means

$$B(x,D) + b(x,D) = \psi(x), \tag{3.2.7}$$

where $\psi(x) = \psi(x,D)$ is the pullback to M of the multiplier operator $\psi(y)$ in local coordinates.

Recall that the η-support of $b(y,\eta)$ is the complement of that for $B(y,\eta)$, and so it must be contained inside of $\mathcal{V}_1$. This means that

$$\int |b_0(x,\xi)|^2 \, d\xi \leq \left(\sup_{x,\xi} |b| \right) |\mathcal{V}_1| \lesssim cT^{-1}, \tag{3.2.8}$$

since we required that $|\mathcal{V}_1| < cT^{-1}$. And so by choosing c small enough, we get $\int |b_0|^2 \, d\xi < T^{-1}$. We can now finally apply these operators to $K_2(x,y)$ to help us finish the proof.

By construction we have that $\psi(z) = \psi^*(z) = 1$ whenever $d_g(x_0, z) \leq \delta_*$, and so if $x, y \in B_{\delta_*}(x_0)$ then we have

$$\begin{aligned} K_2(x,y) &= \psi(x) K_2(x,y) \psi^*(y) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (1-\beta(t))\beta(t/T) \left(\frac{\sin(t/2)}{t/2} e^{-it(\lambda+1/2)} \right) \left(\psi \circ e^{itP} \circ \psi^* \right)(x,y) \, dt, \end{aligned} \tag{3.2.9}$$

where $\left(\psi \circ e^{itP} \circ \psi^* \right)(x,y) = \psi(x) e^{itP}(x,y) \psi^*(y)$ is the Schwartz kernel of the half-wave operator conjugated by ψ. Expanding ψ we arrive at

$$\psi \circ e^{itP} \circ \psi^* = \Big(B(x,D) + b(x,D) \Big) \circ e^{itP} \circ \Big(B^*(x,D) + b^*(x,D) \Big)$$

$$= \quad B(x,D) \circ e^{itP} \circ B^*(x,D) \tag{3.2.10a}$$

$$+ B(x,D) \circ e^{itP} \circ b^*(x,D) \tag{3.2.10b}$$

$$+ b(x,D) \circ e^{itP} \circ B^*(x,D) \tag{3.2.10c}$$

$$+ b(x,D) \circ e^{itP} \circ b^*(x,D). \tag{3.2.10d}$$

Working more generally, we find that if A and B denote arbitrary linear operators, then their compositions with an operator of the form $m(P)$ gives us

$$\begin{aligned} \big((A \circ m(P) \circ B^*) f \big)(x) &= A \Big(m(P) \big(B^* f(x) \big) \Big) \\ &= A \left(\sum_{j=0}^{\infty} m(\lambda_j) \langle B^* f, e_j \rangle \, e_j(x) \right) \end{aligned}$$

$$= A\left(\sum_{j=0}^{\infty} m(\lambda_j)\langle f, Be_j\rangle\, e_j(x)\right)$$

$$= \sum_{j=0}^{\infty} m(\lambda_j)\langle f, Be_j\rangle Ae_j(x)$$

$$= \sum_{j=0}^{\infty} m(\lambda_j)\left(\int_M f(y)\overline{Be_j(y)}\, dV_g(y)\right) Ae_j(x)$$

$$= \int_M \left(\sum_{j=0}^{\infty} m(\lambda_j) Ae_j(x)\overline{Be_j(y)}\right) f(y)\, dV_g(y).$$

This means that the kernel for such an operator is

$$\left(A \circ m(P) \circ B^*\right)(x, y) = \sum_{j=0}^{\infty} m(\lambda_j) Ae_j(x)\overline{Be_j(y)}. \tag{3.2.11}$$

Therefore, by taking $m(P) = e^{itP}$ and substituting $b(x, D)$ or $B(x, D)$ for A or B, along with (3.2.11) and (3.2.10a) through (3.2.10d), we can rewrite (3.2.9) as

$$\psi(x)K_2(x, y)\psi^*(y) = K_{B,B^*}(x, y) + K_{B,b^*}(x, y) + K_{b,B^*}(x, y) + K_{b,b^*}(x, y), \tag{3.2.12}$$

where each

$$K_{A,B^*}(x, y) = \sum_{j=0}^{\infty}\left(\int_{-\infty}^{\infty} X_T(t) e^{-it(\lambda-\lambda_j+1/2)}\, dt\right) Ae_j(x)\overline{Be_j(y)}, \tag{3.2.13}$$

with $X_T(t) = (2\pi)^{-1}(1 - \beta(t))\,\beta(t/T)\frac{\sin(t/2)}{t/2}$. We will need to show that each of these four terms has the appropriate λ-bounds, and will do so by applying two different arguments.

First we handle the three terms that involve either $b(x, D)$ or $b^*(x, D)$ by way of the following lemma:

Lemma 3.4. *Let $B_{\delta_*}(x_0)$ and $K_{b,B^*}(x, y)$, $K_{B,b^*}(x, y)$, $K_{b,b^*}(x, y)$ be defined as above, and suppose that $\varepsilon > 0$. Then there exists $\Lambda > 1$ such that whenever $\lambda \geq \Lambda$,*

$$|K_{b,B^*}(x, y)|, |K_{B,b^*}(x, y)|, |K_{b,b^*}(x, y)| \lesssim \varepsilon\lambda^{n-1} + O(\lambda^{n-2}),$$

for all $x, y \in B_{\delta_*}(x_0)$.

Proof of lemma. Fix $\varepsilon > 0$ and let $x, y \in B_{\delta_*}(x_0)$ be arbitrary. Note that $\widehat{X_T} \in \mathcal{S}(\mathbb{R})$ since $X_T \in C_0^\infty([-T,T]) \subseteq \mathcal{S}(\mathbb{R})$. This fact, combined with (3.2.13), means

$$K_{b,B^*}(x,y) = \sum_{j=0}^{\infty} \widehat{X_T}\left(\lambda - \lambda_j + \tfrac{1}{2}\right) b(x,D)e_j(x)\overline{B(y,D)e_j(y)}, \tag{3.2.14}$$

where $\widehat{X_T}$ is smooth and rapidly decreasing. Similarly for K_{B,b^*} and K_{b,b^*}. This, in turn, means that for any $N \in \mathbb{N}$, there exists some C_N such that

$$\left|\widehat{X_T}\left(\lambda - \lambda_j + \tfrac{1}{2}\right)\right| \lesssim \left(1 + \left|\lambda - \lambda_j + \tfrac{1}{2}\right|\right)^{-N} \leq C_N \left(1 + |\lambda - \lambda_j|\right)^{-N}.$$

Applying Cauchy-Schwarz now gives us

$$\begin{aligned} |K_{b,B^*}(x,y)| &\leq C_N \sum_{j=0}^{\infty} (1 + |\lambda - \lambda_j|)^{-N} \left|be_j(x)\overline{Be_j(y)}\right| \\ &\lesssim \left(\sum_{j=0}^{\infty}(1 + |\lambda - \lambda_j|)^{-N}|be_j(x)|^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty}(1 + |\lambda - \lambda_j|)^{-N}|Be_j(y)|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.2.15}$$

We now borrow a lemma from [11] (cf. lemma 5.2.2) for the following result:

Lemma 3.5. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 2$, and assume the $|\Pi_{per}| = 0$ (the set of periodic geodesics). Then there is a uniform constant $C_M = C(M,g)$, so that whenever $A(x,D) \in \Psi^0_{cl}(M)$ with principal symbol $a(x,\xi)$ we have*

$$\sum_{\lambda_j \in [\lambda, \lambda+1]} |A(x,D)e_j(x)|^2 \leq C_M \left(|g|^{-\frac{1}{2}} \int_{\sum_{j,k} g^{jk}(x)\xi_j\xi_k \leq 1} |a(x,\xi)|^2 \, d\xi\right) \lambda^{n-1} + O_A(\lambda^{n-2}),$$

where $|g| = \det g_{jk}(x)$, and the terms in O_A are bounded by C_A, a constant depending only on A.

We now use this idea of summing over unit-sized bands of frequencies to rewrite each of the series in (3.2.15) as a double sum,

$$\sum_{k=0}^{\infty} \sum_{\lambda_j \in [k,k+1)} (1 + |\lambda - \lambda_j|)^{-N} |Ae_j(z)|^2, \tag{3.2.16}$$

where either $A = b(x,D)$ or $A = B(x,D)$, as in our hypothesis. By fixing the value of $\lambda > 1$, we can then analyze the behavior of (3.2.16) based on the three possible ranges for k, and the fact that $k \leq \lambda_j < k+1$:

- If $0 \le k < \ell$, then $k+1 \le \ell \le \lambda$. Hence,

$$(1+|\lambda-\lambda_j|)^{-N} \le (1+|\lambda-(k+1)|)^{-N}.$$

- If $k > \ell$, then $\lambda < \ell+1 \le k$. Hence,

$$(1+|\lambda-\lambda_j|)^{-N} \le (1+|\lambda-k|)^{-N}.$$

- If $k=\ell$, then $\ell \le \lambda_j < \ell+1$. Hence, whether $\lambda_j < \lambda$ or $\lambda \le \lambda_j$, we have

$$(1+|\lambda-\lambda_j|)^{-N} \le 1.$$

Therefore, for each k, we have

$$k \le \lambda_j < k+1 \qquad \Longrightarrow \qquad (1+|\lambda-\lambda_j|)^{-N} \lesssim (1+|\lambda-k|)^{-N}.$$

Combining this with the results from the lemma, we see that

$$\begin{aligned}\sum_{\lambda_j\in[k,k+1)} (1+|\lambda-\lambda_j|)^{-N}\,|Ae_j(z)|^2 &\lesssim (1+|\lambda-k|)^{-N} \sum_{\lambda_j\in[k,k+1)} |Ae_j(z)|^2 \\ &\lesssim (1+|\lambda-k|)^{-N}\left(\left(|g|^{-\frac{1}{2}}\int |a(x,\xi)|^2\,d\xi\right)k^{n-1}+O_A(k^{n-2})\right)\end{aligned}$$

where either $A=b(x,D)$ or $A=B(x,D)$, and $a(x,\xi)$ is its principal symbol.

Considering (3.2.8), this means that if $A=b(x,D)\in\Psi^0_{cl}(M)$, with principal symbol $b_0(x,\xi)$, then

$$\begin{aligned}\sum_{\lambda_j\in[k,k+1)} |be_j(z)|^2 &\le C_M\left(|g|^{-\frac{1}{2}}\int |b_0(x,\xi)|^2\,d\xi\right)k^{n-1}+O_b(k^{n-2}) \\ &\le C_M\,T^{-1}k^{n-1}+C_b\,k^{n-2},\end{aligned}$$

for some constant C_b depending only on $b(x,D)$. Hence,

$$\sum_{\lambda_j\in[k,k+1)} (1+|\lambda-\lambda_j|)^{-N}\,|be_j(z)|^2 \lesssim (1+|\lambda-k|)^{-N}\left(T^{-1}k^{n-1}+k^{n-2}\right). \tag{3.2.17}$$

Similarly, $A = B(x,D) \in \Psi^0_{cl}(M)$ has a principal symbol with $\operatorname{supp}_\xi B_0(x,\xi) \subset \mathbb{S}^{n-1}$, and so we get

$$\sum_{\lambda_j \in [k,k+1)} |Be_j(z)|^2 \le C_M \left|\mathbb{S}^{n-1}\right| k^{n-1} + C_B\, k^{n-2},$$

for some C_B depending only on $B(x,D)$. Again, combining this result with the conclusions from above gives us

$$\sum_{\lambda_j \in [k,k+1)} (1+|\lambda-\lambda_j|)^{-N} |Be_j(z)|^2 \lesssim (1+|\lambda-k|)^{-N} \left(k^{n-1}+k^{n-2}\right). \tag{3.2.18}$$

It then follows from (3.2.16) through (3.2.18) that

$$\begin{aligned}\sum_{k=0}^{\infty} \sum_{\lambda_j \in [k,k+1)} (1+|\lambda-\lambda_j|)^{-N} |be_j(z)|^2 &\lesssim \sum_{k=0}^{\infty} (1+|\lambda-k|)^{-N} \left(T^{-1}k^{n-1}+k^{n-2}\right) \\ &\lesssim \int_0^\infty (1+|k-\lambda|)^{-N} (T^{-1}k^{n-1}+k^{n-2})\, dk,\end{aligned} \tag{3.2.19}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} \sum_{\lambda_j \in [k,k+1)} (1+|\lambda-\lambda_j|)^{-N} |Be_j(z)|^2 &\lesssim \sum_{k=0}^{\infty} (1+|\lambda-k|)^{-N} \left(k^{n-1}+k^{n-2}\right) \\ &\lesssim \int_0^\infty (1+|k-\lambda|)^{-N} (k^{n-1}+k^{n-2})\, dk.\end{aligned} \tag{3.2.20}$$

All that remains now is to compute the asymptotic behavior of these integrals. First, note that $\int_0^\infty k^{n-1}(1+|k-\lambda|)^{-N}\, dk = \int_0^\lambda k^{n-1}(1+\lambda-k)^{-N}\, dk + \int_\lambda^\infty k^{n-1}(1+k-\lambda)^{-N}\, dk$, where

$$\begin{aligned}\int_0^\lambda k^{n-1}(1+\lambda-k)^{-N}\, dk &= -\int_{\lambda+1}^1 (1+\lambda-r)^{n-1} r^{-N}\, dr \\ &= (\lambda+1)^{n-1} \int_1^{\lambda+1} \left(1-\frac{r}{1+\lambda}\right)^{n-1} r^{-N}\, dr \\ &\le 2^{n-1}\lambda^{n-1} \int_1^\infty r^{-N}\, dr \\ &= O(\lambda^{n-1}).\end{aligned}$$

and

$$\begin{aligned}
\int_{\lambda}^{\infty} k^{n-1}(1+k-\lambda)^{-N}\,dk &= \int_{1}^{\infty} (r+\lambda-1)^{n-1} r^{-N}\,dr \\
&\le \int_{1}^{\infty} (r+\lambda)^{n-1} r^{-N}\,dr \\
&= \lambda^{n-1}\int_{1}^{\infty} \left(\frac{r}{\lambda}+1\right)^{n-1} r^{-N}\,dr \\
&\le \lambda^{n-1}\int_{1}^{\infty} \left(\frac{r}{\lambda}+r\right)^{n-1} r^{-N}\,dr \\
&= \lambda^{n-1}\left(\frac{1}{\lambda}+1\right)^{n-1}\int_{1}^{\infty} r^{-N+n-1}\,dr \\
&\le 2^{n-1}\lambda^{n-1}\int_{1}^{\infty} r^{-N+n-1}\,dr \\
&= O(\lambda^{n-1}).
\end{aligned}$$

Hence $\int_0^\infty k^{n-1}(1+|k-\lambda|)^{-N}\,dk = O(\lambda^{n-1})$. Similarly, $\int_0^\infty k^{n-2}(1+|k-\lambda|)^{-N}\,dk = O(\lambda^{n-2})$. Combining these results with (3.2.15), (3.2.19), and (3.2.20) gives us

$$|K_{b,B^*}(x,y)|, |K_{B,b^*}(x,y)| \lesssim T^{-1/2}\lambda^{n-1} + O(\lambda^{n-2}), \tag{3.2.21}$$

and

$$|K_{b,b^*}(x,y)| \lesssim T^{-1}\lambda^{n-1} + O(\lambda^{n-2}), \tag{3.2.22}$$

meaning that there exists some $\Lambda > 1$ and some $c > 0$ so that if $T > \varepsilon^{-2} \gg 1$ then

$$|K_{b,B^*}(x,y)|, |K_{B,b^*}(x,y)|, |K_{b,b^*}(x,y)| \le c\,\varepsilon\lambda^{n-1}, \quad \text{for all } \lambda > \Lambda,$$

independent of x and y as desired. □

Now that three of the four terms in (3.2.12) have been dealt with, we shift our focus to the final term involving only $B(x,D)$, namely

$$K_{B,B^*}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1-\beta(t))\beta(t/T)\left(\frac{\sin(t/2)}{t/2}e^{-it(\lambda+1/2)}\right)\left(B\circ e^{itP}\circ B^*\right)(x,y)\,dt. \tag{3.2.23}$$

As mentioned in (1.2.8), we know that the Schwartz kernel for the half-wave operator, $e^{itP}(x,y) \in$

$\mathcal{D}'(M \times \mathbb{R} \times M)$, has a wave front set given by

$$\begin{aligned} WF\left(e^{itP}\right) &= \{(x,t,y,\xi,\tau,\eta) : \Phi_t(y,-\eta) = (x,\xi), \tau = p(x,\xi)\} \\ &\subseteq (M \times \mathbb{R} \times M) \times T^*\left(M \times \mathbb{R} \times M\right) \backslash 0 \end{aligned}$$

where $p(x,\xi)$ is the principal symbol of $P = \sqrt{-\Delta_g}$. Moreover, $\operatorname{sing\,supp} e^{itP}$ is the projection of $WF(e^{itP})$ onto $M \times \mathbb{R} \times M$. This means that $(x,t,y) \in \operatorname{sing\,supp} e^{itP}$ only if $\Phi_t(y,-\eta) = (x,\xi)$ for some covectors η, ξ. In other words, if $\Phi_t(y,-\eta) \neq (x,\xi)$ for all covectors η,ξ and all times t, then $e^{itP}(x,y)$ must be smooth in a neighborhood of (x,t,y).

Recall from (3.2.2) that $B(y,\eta)$ is simply a smooth cutoff function, and hence the corresponding pseudodifferential operator $B(x,D)$ will be smoothing, so as not to introduce any new singular points to $(B \circ e^{itP} \circ B^*)(x,y)$. Also, by (3.2.3), we know that if $x, y \in \operatorname{supp}_M B$, then the return time any "good" geodesic leaving from these points is at least T, which will be nullified by the term $\beta(t/T)$ in (3.2.23). Therefore, since geodesic flow from y *cannot* flow back to x in time $1 \leq |t| \leq T$, we must have that $(x,t,y) \notin WF\left(e^{itP}\right)$, and so $(B \circ e^{itP} \circ B^*)(x,y)$ must be smooth in x, y, and t. This allows us to rewrite (3.2.23) as

$$K_{B,B^*}(x,y) = \frac{1}{2\pi} \int_{\infty}^{\infty} (1-\beta(t))\beta(t/T) e^{-it\lambda} F(t,x,y)\, dt,$$

where $F(t,x,y) \in C^\infty(\mathbb{R} \times \tilde{\Omega} \times \tilde{\Omega})$.

From here all that is left is to use the standard "non-stationary phase" argument, since $\lambda \neq 0$, that $L^N[e^{it\lambda}] = e^{it\lambda}$ for all $N \in \mathbb{N}$, where

$$L := \frac{1}{(i\lambda)^N} \left(\frac{d}{dt}\right)^N.$$

This implies that

$$\sup_{x,y \in B_{\delta_*}(x_0)} |K_{B,B^*}(x,y)| = \frac{1}{2\pi\lambda^N} \left| \int_{-\infty}^{\infty} (1-\beta(t))\beta(t/T) \frac{d^N}{dt^N}\left[e^{it\lambda}\right] F(t,x,y)\, dt \right| \tag{3.2.24}$$

$$\lesssim \lambda^{-N} \int_{-\infty}^{\infty} \left| \frac{d^N}{dt^N}\Big[(1-\beta(t))\beta(t/T)F(t,x,y)\Big] \right| dt. \tag{3.2.25}$$

But $(1-\beta(t))\beta(t/T)F(t,x,y) \in C_0^\infty([1,T])$, and so the integrand above has finite $L^\infty(\mathbb{R})$ norm, meaning the integral is convergent. Moreover, since $F(t,x,y)$ and all of its derivatives are continuous on $\overline{B_{\delta_*}(x_0)} \Subset B_{\delta'_*}(x_0)$, then $\sup_{x,y\in B_{\delta_*}(x_0)} |K_{B,B^*}(x,y)| \lesssim C_{\delta_*}\lambda^N$, meaning that $|K_{B,B^*}| = O(\lambda^{-N})$, for any $N \in \mathbb{N}$, and hence contributes only negligibly to our overall results.

We have now finished proving the desired asymptotic bounds for each part of $K_2(x,y)$ and thus, in conclusion, there exists a $\delta_* > 0$ and $\Lambda_* > 1$ so that if $x, y \in M$ with $d_g(x,y) < \delta_*$, then $|K_2(x,y)| \lesssim \varepsilon\lambda^{n-1} + O(\lambda^{n-2})$, for all $\lambda > \Lambda_*$. $\square$

3.3 Large Time Estimates

Proof of Theorem 1.1 (large time). We now finally focus our attention on the large time estimates in (2.0.2c), regarding the term K_3. Recall that $\varepsilon > 0$ is fixed, and that

$$\begin{aligned}
K_3(x,y) &= (2\pi)^{-1}\int_{-\infty}^{\infty} \big(1-\beta(t/T)\big)\,\frac{\sin(t/2)}{t/2}\,e^{-it(\lambda+1/2)}\,e^{itP}(x,y)\,dt \\
&= (2\pi)^{-1}\sum_{j=0}^{\infty}\left(\int_{-\infty}^{\infty}\big(1-\beta(t/T)\big)\frac{\sin(t/2)}{t/2}e^{-it(\lambda-\lambda_j+1/2)}\,dt\right) e_j(x)\overline{e_j(y)} \\
&= \sum_{j=0}^{\infty}\widehat{X_T}\big(\lambda-\lambda_j+\tfrac{1}{2}\big)e_j(x)\,\overline{e_j(y)},
\end{aligned} \tag{3.3.1}$$

where $X_T(t) = (2\pi)^{-1}\big(1-\beta(t/T)\big)\frac{\sin(t/2)}{t/2} \in C(\mathbb{R})\cap L^2(\mathbb{R})$. We wish to show that the kernel $K_3(x,y)$ also has bounds that are $o(\lambda^{n-1})$, but to do that we first need to prove that $\widehat{X_T}(\tau) = O\left((1+|\tau|)^{-N}\right)$ for N sufficiently large.

Consider

$$\begin{aligned}
\widehat{X_T}(\tau) &= (2\pi)^{-1}\int_{-\infty}^{\infty}\big(1-\beta(t/T)\big)\frac{\sin(t/2)}{t/2}e^{-it\tau}\,dt \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty}T\big(1-\beta(s)\big)\frac{\sin(sT/2)}{sT/2}e^{-isT\tau}\,ds \\
&= -i(2\pi)^{-1}\int_{-\infty}^{\infty}\left(\frac{1-\beta(s)}{s}\right)\big(e^{isT/2}-e^{-isT/2}\big)e^{-isT\tau}\,ds \\
&= C_{\pm}\sum_{\pm}\int_{-\infty}^{\infty}\left(\frac{1-\beta(s)}{s}\right)e^{-isT(\tau\pm 1/2)}\,ds,
\end{aligned} \tag{3.3.2}$$

where the integral in the last line is understood either in the sense of distributions, or as the Fourier transform of an $L^2(\mathbb{R})$ function. We now focus on the following lemma:

Lemma 3.6. *Define the function*

$$I(\theta) = \int_{-\infty}^{\infty} \left(\frac{1-\beta(s)}{s} \right) e^{-is\theta}\, ds, \tag{3.3.3}$$

Where $\beta(s)$ is an even, smooth bump function with $\text{supp}\beta \subseteq [-1,1]$*, and a constant value of 1 on the interval* $(-1/2, 1/2)$*. Then $I(\theta) = O(1+|\theta|)^{-N}$ for any $N \in \mathbb{N}$.*

Proof of Lemma. We handle (3.3.3) by breaking it up into a few separate cases:

- First, suppose that $|\theta| \geq 1$. Following a typical non-stationary phase argument, we note that for any $N \in \mathbb{N}$, we have $\frac{d^N}{ds^N} e^{-is\theta} = (-i\theta)^N e^{-is\theta}$, or equivalently that $(\frac{i}{\theta}\frac{d}{ds})^N e^{-is\theta} = e^{-is\theta}$. Integration by parts, and the fact that $(1-\beta(s))/s \to 0$ as $s \to \infty$, while $e^{-is\theta}$ remains bounded, allows us to rewrite the integral as

$$\begin{aligned} |I(\theta)| &= \left| \int_{-\infty}^{\infty} \left(\frac{1-\beta(s)}{s} \right) \left(\frac{i}{\theta}\frac{d}{ds} \right)^N \left[e^{-is\theta} \right] ds \right| \\ &= |\theta|^{-N} \left| \int_{-\infty}^{\infty} \frac{d^N}{ds^N} \left[\frac{1-\beta(s)}{s} \right] e^{-is\theta}\, ds \right| \\ &\leq |\theta|^{-N} \int_{-\infty}^{\infty} \left| \frac{d^N}{ds^N} \left[\frac{1-\beta(s)}{s} \right] \right| ds \\ &\leq C_N |\theta|^{-N}, \end{aligned} \tag{3.3.4}$$

  where

$$\left| \frac{d^N}{ds^N} \left(\frac{1-\beta(s)}{s} \right) \right| \lesssim \frac{1}{1+|s|^{N+1}}$$

  since $1-\beta(s) \equiv 0$ on $(-1/2, 1/2)$, and $\frac{d^N}{ds^N}(1-\beta(s)) \in L^\infty(\mathbb{R})$ for all $N \in \mathbb{N}$. Hence for large values of θ we have see that $I(\theta)$ is rapidly decreasing. Now all that's left is to prove that $I(\theta)$ is bounded near $\theta = 0$. To do this we will consider the integral in (3.3.3) as two pieces: one for $|s| \leq |\theta|^{-1}$, and one for $|s| > |\theta|^{-1}$,

$$A + B = \int_{|s|\leq|\theta|^{-1}} \left(\frac{1-\beta(s)}{s} \right) e^{-is\theta}\, ds + \int_{|s|>|\theta|^{-1}} \left(\frac{1-\beta(s)}{s} \right) e^{-is\theta}\, ds. \tag{3.3.5}$$

- Suppose that $0<|\theta|<1$ and $|s|\le|\theta|^{-1}$. Then

$$|A| = \left|\int_{|s|\le|\theta|^{-1}} \left(\frac{1-\beta(s)}{s}\right) e^{-is\theta}\, ds\right|$$
$$= \left|\int_{-|\theta|^{-1}}^{|\theta|^{-1}} (1-\beta(s))\, \frac{\sin(s\theta)}{s}\, ds\right| \tag{3.3.6}$$
$$\le |\theta| \int_{-|\theta|^{-1}}^{|\theta|^{-1}} |1-\beta(s)|\, ds \tag{3.3.7}$$
$$\le 2, \tag{3.3.8}$$

where (3.3.6) follows from the fact that $\beta(s)$ is an even function, (3.3.7) follows from the inequality $|\sin(s\theta)/s| \le |\theta|$ for all s, and (3.3.8) follows from $||1-\beta(s)||_\infty = 1$.

- Now suppose that $0<|\theta|<1$ and focus, at first, on $s>|\theta|^{-1}$:

$$\left|\int_{|\theta|^{-1}}^{\infty} \left(\frac{1-\beta(s)}{s}\right) e^{-is\theta}\, ds\right| = \left|\frac{i}{\theta}\int_{|\theta|^{-1}}^{\infty} \left(\frac{1-\beta(s)}{s}\right) \frac{d}{ds}[e^{-is\theta}]\, ds\right|$$
$$= \frac{1}{|\theta|}\left|\left(\frac{1-\beta(s)}{s}\right) e^{-is\theta}\Big|_{|\theta|^{-1}}^{\infty} - \int_{|\theta|^{-1}}^{\infty} \frac{d}{ds}\left[\frac{1-\beta(s)}{s}\right] e^{-is\theta}\, ds\right|$$
$$\le \frac{1}{|\theta|}\left(|\theta| + \int_{|\theta|^{-1}}^{\infty} \frac{|\beta(s)-1-s\beta'(s)|}{|s|^2}\, ds\right) \tag{3.3.9}$$
$$\le 1 + \frac{C}{|\theta|}\int_{|\theta|^{-1}}^{\infty} \frac{1}{|s|^2}\, ds$$
$$= 1 + C \tag{3.3.10}$$

where in (3.3.9) we use the fact that $\beta(|\theta|^{-1}) = 0$ since $\mathrm{supp}\beta \subseteq (-1,1)$. The same computation holds for $s < -|\theta|^{-1}$, and so we get $|B| = O(1)$.

- And finally, since $I(\theta) = \mathcal{F}\left[(1-\beta(s))/s\right](\theta)$, with $(1-\beta(s))/s \in L^2(\mathbb{R})$ an odd function, then $I(0) = 0$.

All together (3.3.4), (3.3.8), and (3.3.10) all imply that $I(\theta)$ is both bounded and rapidly decreasing, and hence $|I(\theta)| \lesssim (1+|\theta|)^{-N}$ for any $N\in\mathbb{N}$.

□

We can now apply the lemma to (3.3.1) and (3.3.2) with $\theta = \theta_\pm(\lambda) = T\left(\left(\lambda-\lambda_j+\frac{1}{2}\right) \pm \frac{1}{2}\right)$. Moreover, since

$$\theta_+(\lambda) = T(\lambda-\lambda_j+1) = \theta_-(\lambda+1),$$

then both $I(\theta_+)$ and $I(\theta_-)$ are $O\left(\left(1+T\,|\lambda-\lambda_j|\,\right)^{-N}\right)$ which gives us that

$$\widehat{X_T}\left(\lambda-\lambda_j+\tfrac{1}{2}\right)\lesssim\left(1+T\,|\lambda-\lambda_j|\,\right)^{-N},\quad \text{for all } N\geq 1. \tag{3.3.11}$$

Now, by applying Cauchy-Schwarz to (3.3.1), along with (3.3.11), we see that

$$\begin{aligned}|K_3(x,y)| &\leq \left(\sum_{j=0}^{\infty}\left|\widehat{X_T}\left(\lambda-\lambda_j+\tfrac{1}{2}\right)\right||e_j(x)|^2\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty}\left|\widehat{X_T}\left(\lambda-\lambda_j+\tfrac{1}{2}\right)\right||e_j(y)|^2\right)^{\frac{1}{2}}\\ &\lesssim \left(\sum_{j=0}^{\infty}\left(1+T\,|\lambda-\lambda_j|\,\right)^{-N}|e_j(x)|^2\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty}\left(1+T\,|\lambda-\lambda_j|\,\right)^{-N}|e_j(y)|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Recall that $T\gg 1$ is fixed, and that x,y are in a neighborhood of some fixed $x_0\in M$. Focusing for a moment on one of the series above, for a fixed $\lambda>1$ we re-express the summation over j as a double-sum over regions of size comparable to $T^{-1}\ll 1$:

$$\sum_{k=0}^{\infty}\sum_{\lambda_j\in I_k}\left(1+T|\lambda-\lambda_j|\right)^{-N}|e_j(z)|^2, \tag{3.3.12}$$

where $z\in\mathcal{N}_{x_0}$ and I_k is the region in $\mathbb{R}$ where the distance from λ is between k/T and $(k+1)/T$. In other words,

$$\begin{aligned}I_k &= \left\{r\in\mathbb{R}:\tfrac{k}{T}\leq|\lambda-r|\leq\tfrac{k+1}{T}\right\}\\ &=\left[\lambda-\tfrac{k+1}{T},\lambda-\tfrac{k}{T}\right]\cup\left[\lambda+\tfrac{k}{T},\lambda+\tfrac{k+1}{T}\right]\end{aligned}$$

the union of two intervals of size T^{-1}. In particular, this means that if $\lambda_j\in I_k$, then

$$|\lambda-\lambda_j|\approx kT^{-1}. \tag{3.3.13}$$

We now appeal to a theorem from [11] (cf. theorem 5.4.1). This theorem, along with [11] (cf. 5.4.6) provides for us the following lemma:

Lemma 3.7. *Fix a Riemannian manifold (M,g) of dimension $n\geq 2$, and assume that every $x\in M$ is non self-focal. If $x_0\in M$, and $T\gg 1$, then there exists a neighborhood of $\mathcal{N}_{x_0}$ of x_0 (depending on T), a number $\Lambda<\infty$ (depending on T and $\mathcal{N}_{x_0}$), and a constant C (depending* *only on the manifold (M,g)) such that*

$$\sum_{\lambda_k\in[\lambda,\lambda+T^{-1}]} |e_k(x)|^2 \le CT^{-1}\lambda^{n-1}, \quad x\in\mathcal{N}_{x_0}, \quad \lambda\ge\Lambda. \tag{3.3.14}$$

Applying (3.3.14) from the lemma directly, we see that for all $z\in\mathcal{N}_{x_0}$ we have

$$\sum_{\lambda_j\in I_k} |e_j(z)|^2 \lesssim T^{-1}(\lambda+k/T)^{n-1}, \quad \lambda>\Lambda. \tag{3.3.15}$$

Hence, if we now combine (3.3.12), (3.3.13), and (3.3.15), we finally arrive at the following bounds,

$$\begin{aligned}|K_3(x,y)| &\lesssim \left(\left(\sum_{k=0}^{\infty}\big(1+T(kT^{-1})\big)^{-N} T^{-1}(\lambda+k/T)^{n-1}\right)^{1/2}\right)^2 \\ &\lesssim T^{-1}\int_0^\infty (1+k)^{-N}(\lambda+k/T)^{n-1}\,dk, \end{aligned}\tag{3.3.16}$$

when $\lambda>\Lambda$, as in the lemma. To finish the proof, we simply need to evaluate (3.3.16) in two parts:

- Suppose that $0\le k\le\lambda$. Then $1\le 1+\frac{k}{\lambda T}\le 1+T^{-1}<2$, and

$$\begin{aligned}\int_0^\lambda (1+k)^{-N}(\lambda+k/T)^{n-1}\,dk &= \lambda^{n-1}\int_0^\lambda (1+k)^{-N}(1+\tfrac{k}{\lambda T})^{n-1}\,dk \\ &\le 2^{n-1}\lambda^{n-1}\int_0^\lambda (1+k)^{-N}\,dk \\ &\lesssim \lambda^{n-1}\left(1-\lambda^{-(N-1)}\right) \\ &= O(\lambda^{n-1}), \quad \text{if } N>n. \end{aligned}\tag{3.3.17}$$

- Suppose that $\lambda<k$. Then

$$\begin{aligned}\int_\lambda^\infty (1+k)^{-N}(\lambda+k/T)^{n-1}\,dk &= \frac{1}{T^{n-1}}\int_\lambda^\infty (1+k)^{-N}(\lambda T+k)^{n-1}\,dk \\ &\le \frac{1}{T^{n-1}}\int_\lambda^\infty (1+k)^{-N}(kT+k)^{n-1}\,dk \\ &= \left(\frac{T+1}{T}\right)^{n-1}\int_\lambda^\infty (1+k)^{-N}k^{n-1}\,dk \\ &\le C\int_\lambda^\infty k^{-N+n+1}\,dk \\ &\le C_N\lambda^{n-N} \\ &= O(1), \quad \text{if } N>n. \end{aligned}\tag{3.3.18}$$

So finally, based on (3.3.16), (3.3.17), and (3.3.18), we can conclude that for $\lambda > \Lambda$ as in the above lemma, we have

$$|K_3(x,y)| \lesssim T^{-1}\lambda^{n-1} < \varepsilon\lambda^{n-1},$$

if T is chosen large enough. To make everything precise, first we choose T such that $T^{-1} < \varepsilon$. Next, we appeal to the lemma which gives us $\mathcal{N}_{x_0}$, a neighborhood of x_0, and Λ dependent on T and $\mathcal{N}_{x_0}$. So finally, we pick δ_0 to be small enough so that $B_{\delta_0}(x_0) \subseteq \mathcal{N}_{x_0}$. Hence $K_3(x,y) = o(\lambda^{n-1})$ for any $x, y \in B_{\delta_0}(x_0)$, and any $\lambda > \Lambda$ as desired.

$\square$

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